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Preprints No. 305
(Technical Report No. 2003/9)

**INSTITUTE OF
MATHEMATICS AND INFORMATICS
2003**

CONVERGENCE RATES IN THE LAW OF LARGE NUMBERS FOR ARRAYS OF BANACH SPACE VALUED RANDOM ELEMENTS

By Tibor Tómács (Eger)

1. Introduction

Several papers are devoted to the study of convergence rates in the law of large numbers. The well-known theorem of Baum and Katz [2] states the following. Let $X_1, X_2, \dots$ be independent identically distributed random variables with $\mathbf{E}X_k = 0$ if $\mathbf{E}|X_k| < \infty$. Let $t > 0$, $r \geq 1$ and $2r > t$. Then $\mathbf{E}|X_k|^t < \infty$ if and only if

$$\sum_{n=1}^{\infty} n^{r-2} \mathbf{P}(|S_n| > \varepsilon n^{r/t}) < \infty \quad \text{for all} \quad \varepsilon > 0.$$

Earlier versions (special cases) of this theorem were obtained by Hsu and Robbins [10], Erdős [3], [4] and Spitzer [14]. The result was extended to Banach space valued random variables (Jain [13], Woyczyński [15]), to arrays of random variables (Hu, Móricz and Taylor [11], Gut [8]). See Ahmed, Antonini and Volodin [1] for an overview of the recent progress.

Instead of the whole sequence S_n, one can study the subsequence S_{k_n}. For arrays of random variables it means the study of general arrays (see Gut [8], Fazekas [6], Hu et al. [12]). The aim of this note is to show that an appropriate version of the classic result of Jain [13] (Theorem 3.3) implies several theorems on general arrays.

Throughout the paper we study Banach space valued random variables. However, some of our results are new for real variables, too. In Section 2 we introduce notation. The main results are in Section 3. Theorem 3.1 is a generalization of Theorem 3.3 of Jain [13]. The idea in Theorem 3.1 is the following. When we apply Hoffmann–Jørgensen's inequality, we use two different functions to obtain upper bounds for the two terms in the inequality. The theorem obtained seems to be difficult, but when we choose appropriate weight functions we can obtain several known theorems for general arrays like $X_{n1}, \dots, X_{nk_n}$. Corollaries 3.2 and 3.3 are versions of Theorem 6.2 of Fazekas [6] and Corollary 4.1 of Hu et al. [12], respectively. In Section 4 we give the proofs. In Section 5 we specialize our result for Banach spaces with some geometric property. Then we obtain new proofs for results in Fazekas [6] and Hu et al. [12]. We do not list all special cases of our result, we only refer to the literature.

2. Notation

Let $\mathbf{N}$ be the set of the positive integers, $\mathbf{R}$ the set of real numbers, $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$, where $a, b \in \mathbf{R}$. Denote by R_f the range of the function f and by $f \circ g$ the composite function of functions f and g.

Let Φ_0 denote the set of functions $f\colon [0, \infty) \to [0, \infty)$, that are nondecreasing. A function $f \in \Phi_0$ is said to be an *Orlicz function* if it is continuous, convex, unbounded, $f(0) = 0$ and $f(t) > 0$ for $t > 0$.

A function $f \in \Phi_0$ is said to satisfy the Δ_2-condition ($f \sim \Delta_2$) if there exists a constant $c > 0$, such that

$$f(2t) \le cf(t) \tag{2.1}$$

for all $t > 0$. It is clear that $f \sim \Delta_2$ iff for every fixed $k > 1$, there exists a constant $c > 1$, such that

$$f(kt) \le cf(t) \quad \text{for all} \quad t > 0. \tag{2.2}$$

A function $f \in \Phi_0$ is said to satisfy the Δ_2^0-condition ($f \sim \Delta_2^0$) if there exist constants $c > 0$ and $t_0 > 0$, such that (2.1) is satisfied for all $0 < t \le t_0$.

Throughout the paper let $\{k_n, n \in \mathbf{N}\}$ be a strictly increasing sequence of positive integers. Following Gut [7], introduce the functions ψ and M_r with

$$\psi(t) = \mathrm{Card}\{n \in \mathbf{N} : k_n \le t\} \quad \text{for} \quad t > 0 \quad \text{and} \quad \psi(0) = 0,$$

and

$$M_r(t) = \sum_{i=1}^{[t]} k_i^{r-1} \quad \text{if} \quad t \ge 1 \quad \text{and} \quad M_r(t) = k_1^{r-1} \quad \text{if} \quad 0 \le t < 1,$$

where $r \in \mathbf{R}$, $\mathrm{Card}A$ is the cardinality of the set A and $[.]$ denotes the integer function. Let $M = M_2$.

Let B be a real separable Banach space with norm $\|.\|$ and zero element $\mathbf{0}$. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a fixed probability space. $X\colon \Omega \to B$ is called a B-valued random variable (r.v.), if $\{\omega \in \Omega : X(\omega) \in A\} \in \mathcal{F}$ for all $A \in \mathcal{B}(B)$, where $\mathcal{B}(B)$ denotes Borel σ-field of B. If $\mathbf{E}\,\|X\| < \infty$ then $\mathbf{E}X$ stands for the Bochner integral of B-valued r.v. X.

X is *symmetric* if X and $-X$ have same distribution. The *symmetrization procedure* consists in assigning to the r.v. X the *symmetrized* r.v. $X^* = X - X'$, where X' is independent of X and has the same distribution. Then

$$\mathbf{P}(\|X'\| < t)\mathbf{P}(\|X\| > 2t) \le \mathbf{P}(\|X^*\| > t) \le 2\mathbf{P}(\|X - b\| > t/2) \tag{2.3}$$

for all $t \ge 0$ and $b \in B$.

Let $\{X_{nk}, n \in \mathbf{N}, k = 1, \dots, k_n\}$ be an array of B-valued r.v.'s. It is rowwise independent, if $X_{n1}, \dots, X_{nk_n}$ are independent r.v.'s for all $n \in \mathbf{N}$. Let $S_{k_n} = \sum_{k=1}^{k_n} X_{nk}$.

Definition 2.1. (Gut [8]). We say that the array $\{X_{nk}, n \in \mathbf{N}, k = 1, \dots, k_n\}$ is *weakly mean dominated* (w.m.d.) by the r.v. X, if for some $\gamma > 0$,

$$\frac{1}{k_n} \sum_{k=1}^{k_n} \mathbf{P}(\|X_{nk}\| > t) \le \gamma \mathbf{P}(|X| > t) \quad \text{for all} \quad t \ge 0 \quad \text{and} \quad n \in \mathbf{N}. \tag{2.4}$$

Remark 2.2. Let ξ be a real valued r.v. and $t > 0$ fixed. Since $\bigcap_{m=1}^{\infty}\{\xi > t - 1/m\} = \{\xi \geq t\}$, hence by continuity of probability we get, that if $\{X_{nk}, n \in \mathbf{N}, k = 1, \ldots, k_n\}$ is w.m.d. by the r.v. X, then

$$\frac{1}{k_n}\sum_{k=1}^{k_n} \mathbf{P}(\|X_{nk}\| \geq t) \leq \gamma \mathbf{P}(|X| \geq t) \quad \text{for all} \quad t > 0 \quad \text{and} \quad n \in \mathbf{N}. \tag{2.5}$$

3. A general convergence rate theorem

Theorem 3.1. *Let $\{X_{nk}, n \in \mathbf{N}, k = 1, \ldots, n\}$ be an array of rowwise independent B-valued r.v.'s which is w.m.d. by the r.v. X. We assume that there exists a sequence $\{\gamma_n, n \in \mathbf{N}\}$ of positive real numbers such that $\{\|S_n\|/\gamma_n, n \in \mathbf{N}\}$ is bounded in probability. Let $\alpha, \vartheta, \varphi \in \Phi_0$, α is not bounded, $\vartheta, \varphi \sim \Delta_2$, $\vartheta \not\equiv 0$ and*

$$\beta(n) = \varphi(\alpha(n+1)) - \varphi(\alpha(n)), \quad n = 0, 1, 2, \ldots.$$

We assume that

$$\mathbf{E}\,\varphi(|X|) < \infty, \quad \mathbf{E}\,\vartheta(|X|) < \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{\alpha(n)}{\gamma_n} = \infty.$$

Let

$$\mu(n) = \beta(n-1) \quad \text{for all} \quad n \in \mathbf{N}$$

or

$$\mu(n) = \beta(n) \quad \text{for all} \quad n \in \mathbf{N}.$$

In case $\mu = \beta$ assume that there exists a constant $c > 0$ such that for $n \in \mathbf{N}$ large enough

$$c\beta(n) \leq \beta(n-1). \tag{3.1}$$

Let $n_0 \in \mathbf{N}$ such that $\vartheta(\alpha(n)) > 0$ for all $n \geq n_0$. If there exist $j \in \mathbf{N}$ and $r > 0$ such that

$$\sum_{n=n_0}^{\infty} \frac{\mu(n)}{n} \left(\frac{rn + \vartheta(\gamma_n)}{\vartheta(\alpha(n))} \right)^{2^j} < \infty \tag{3.2}$$

then

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \mathbf{P}\big(\|S_n\| > \varepsilon\alpha(n)\big) < \infty \quad \text{for all} \quad \varepsilon > 0. \tag{3.3}$$

The following corollary is a generalization of Theorem 6.2 of Fazekas [6].

Corollary 3.2. *Let $M \circ \psi \sim \Delta_2$, $r, s, t > 0$, $rs > t$. In case $r > 2$ assume that $\{M(n)/M(n-1), n \in \mathbf{N}\}$ is bounded. Let $\{X_{nk}, n \in \mathbf{N}, k = 1, \ldots, k_n\}$ be an array* *of rowwise independent B-valued r.v.'s which is w.m.d. by the r.v. X. Assume that* $\{\|S_{k_n}\|/k_n^{1/s}, n \in \mathbf{N}\}$ *is bounded in probability. If*

$$\mathbf{E}M^{r/2}\Big(\psi\left(|X|^{t/r}\right)\Big) < \infty \quad \textit{and} \quad \mathbf{E}\,|X|^s < \infty$$

then

$$\sum_{n=1}^{\infty}\big(M(n)\big)^{r/2-1}\mathbf{P}\left(\|S_{k_n}\| > \varepsilon k_n^{r/t}\right) < \infty \quad \textit{for all} \quad \varepsilon > 0.$$

We remark that the boundedness of $\{M(n)/M(n-1), n \in \mathbf{N}\}$ is equivalent to (5.5).

The following corollary is a version of Corollary 4.1 of Hu et al. [12].

Corollary 3.3. *Let $r \in \mathbf{R}$, $0 < t < s$ and $M_r \circ \psi \sim \Delta_2$. Let $\{X_{nk}, n \in \mathbf{N}, k = 1, \dots, k_n\}$ be an array of rowwise independent B-valued r.v.'s which is w.m.d. by the r.v. X. We assume that $\{\|S_{k_n}\|/k_n^{1/s}, n \in \mathbf{N}\}$ is bounded in probability. If*

$$\mathbf{E}M_r\Big(\psi\big(|X|^t\big)\Big) < \infty \quad \textit{and} \quad \mathbf{E}\,|X|^s < \infty$$

then

$$\sum_{n=1}^{\infty} k_n^{r-2}\mathbf{P}\left(\|S_{k_n}\| > \varepsilon k_n^{1/t}\right) < \infty \quad \textit{for all} \quad \varepsilon > 0.$$

4. Proofs

We start with some preliminary results. The following lemma is a version of Lemma 2.2 of Jain [13].

Lemma 4.1. *Let X be a r.v., $\varphi, \alpha \in \Phi_0$, $\beta(n) = \varphi(\alpha(n+1)) - \varphi(\alpha(n))$, $n = 0, 1, 2, \dots$. If $\mathbf{E}\,\varphi(|X|) < \infty$, then*

$$\sum_{n=1}^{\infty}\beta(n-1)\mathbf{P}(|X| \geq \alpha(n)) < \infty.$$

Proof. With notation $\Theta_n = \varphi(\alpha(n))$ we have

$$\begin{aligned}
\mathbf{E}\,\varphi(|X|) &\geq \sum_{i=1}^{\infty}\Theta_i\mathbf{P}\Big(\Theta_i \leq \varphi(|X|) < \Theta_{i+1}\Big)\\
&\geq \sum_{i=1}^{\infty}\sum_{n=1}^{i}\beta(n-1)\mathbf{P}\Big(\Theta_i \leq \varphi(|X|) < \Theta_{i+1}\Big)\\
&= \sum_{n=1}^{\infty}\beta(n-1)\sum_{i=n}^{\infty}\mathbf{P}\Big(\Theta_i \leq \varphi(|X|) < \Theta_{i+1}\Big)\\
&\geq \sum_{n=1}^{\infty}\beta(n-1)\mathbf{P}(|X| \geq \alpha(n)).
\end{aligned}$$

□

The following lemma is due to Hoffmann-Jørgensen [9] and Jain [13].

Lemma 4.2. *Let $X_1, \dots, X_n$ be B-valued, independent, symmetric r.v.'s and $j \in \mathbf{N}$. Then there exists $A_j, B_j \geq 0$ such that*

$$\mathbf{P}\left(\left\|\sum_{k=1}^{n} X_k\right\| > 3^j t\right) \leq A_j \mathbf{P}\left(\max_{1\leq k\leq n} \|X_k\| > t\right) + B_j \mathbf{P}^{2^j}\left(\left\|\sum_{k=1}^{n} X_k\right\| > t\right)$$

for all $t \geq 0$, where A_j and B_j depend only on j. ($A_1 = 1$, $B_1 = 4$.)

The following lemma is a generalization of Theorem 3.1 of Jain [13] and Lemma 2.6 of Fazekas [6].

Lemma 4.3. *Let $\{X_{nk}, n \in \mathbf{N}, k = 1, \dots, k_n\}$ be an array of rowwise independent, symmetric B-valued r.v.'s and $\{\gamma_n, n \in \mathbf{N}\}$ be a sequence of positive real numbers. If $\{\|S_{k_n}\|/\gamma_{k_n}, n \in \mathbf{N}\}$ is bounded in probability, $\vartheta \in \Phi_0$ and $\vartheta \sim \Delta_2$, then there exist constants $a, b > 0$ such that*

$$\mathbf{E}\,\vartheta(\|S_{k_n}\|) \leq a\mathbf{E}\,\vartheta\left(\max_{1\leq k\leq k_n} \|X_{nk}\|\right) + b\vartheta(\gamma_{k_n}) \quad \textit{for all} \quad n \in \mathbf{N}.$$

Proof. Let $N_{k_n} = \max_{1\leq k\leq k_n} \|X_{nk}\|$. By $\vartheta \in \Phi_0$ and Lemma 4.2, we have for all $x \geq 0$ and $n \in \mathbf{N}$

$$\begin{aligned}\mathbf{P}(\vartheta(\|S_{k_n}\|/3) > \vartheta(x)) &\leq \mathbf{P}(\|S_{k_n}\|/3 > x)\\ &\leq \mathbf{P}(N_{k_n} > x) + 4\mathbf{P}^2(\|S_{k_n}\| > x)\\ &\leq \mathbf{P}(\vartheta(N_{k_n}) \geq \vartheta(x)) + 4\mathbf{P}^2(\vartheta(\|S_{k_n}\|) \geq \vartheta(x)).\end{aligned}$$

Hence

$$\mathbf{P}(\vartheta(\|S_{k_n}\|/3) > t) \leq \mathbf{P}(\vartheta(N_{k_n}) \geq t) + 4\mathbf{P}^2(\vartheta(\|S_{k_n}\|) \geq t) \tag{4.1}$$

for all $t \in R_\vartheta$ and $n \in \mathbf{N}$. Assume that $t \in (\vartheta(0), \sup R_\vartheta) \cap \overline{R_\vartheta}$. Then there exists $a \geq 0$, so that $\lim_{x\to a-0} \vartheta(x) < t < \lim_{x\to a+0} \vartheta(x)$. (Let $\lim_{x\to 0-0} \vartheta(x) = \vartheta(0)$.) If $\vartheta(a) < t$, then $\bigcup_{m=1}^{\infty}\{y : \vartheta(y) > \vartheta(a + 1/m)\} = \{y : \vartheta(y) > t\}$ and $\bigcup_{m=1}^{\infty}\{y : \vartheta(y) \geq \vartheta(a + 1/m)\} = \{y : \vartheta(y) \geq t\}$. On the other hand, if $\vartheta(a) > t$, then $\bigcap_{m=1}^{\infty}\{y : \vartheta(y) > \vartheta(a - 1/m)\} = \{y : \vartheta(y) > t\}$ and $\bigcap_{m=1}^{\infty}\{y : \vartheta(y) \geq \vartheta(a - 1/m)\} = \{y : \vartheta(y) \geq t\}$. Hence, using continuity of probability and (4.1), we have that (4.1) is true in this case as well. If $0 \leq t \leq \vartheta(0)$ or $t \geq \sup R_\vartheta$, then (4.1) is obvious. Now, applying $\vartheta \sim \Delta_2$, there exists a constant $c > 1$ such that

$$\mathbf{P}(\vartheta(\|S_{k_n}\|) > ct) \leq \mathbf{P}(\vartheta(N_{k_n}) \geq t) + 4\mathbf{P}^2(\vartheta(\|S_{k_n}\|) \geq t) \tag{4.2}$$

for all $t \geq 0$. Integrating with respect to t gives

$$\frac{1}{c}\mathbf{E}\,\vartheta(\|S_{k_n}\|) \leq \mathbf{E}\,\vartheta(N_{k_n}) + 4\int_0^\infty \mathbf{P}^2(\vartheta(\|S_{k_n}\|) > t)\,dt. \tag{4.3}$$

Since $\{\|S_{k_n}\|/\gamma_{k_n}, n \in \mathbf{N}\}$ is bounded in probability and $\vartheta \sim \Delta_2$ there exist constants $A_1, A > 0$ such that

$$\mathbf{P}\left(\|S_{k_n}\| \geq A_1\gamma_{k_n}\right) < \frac{1}{8c} \quad \text{and} \quad \vartheta(A_1\gamma_{k_n}) \leq A\vartheta(\gamma_{k_n})$$

for all $n \in \mathbf{N}$. Hence we have

$$\mathbf{P}\big(\vartheta(\|S_{k_n}\|) > A\vartheta(\gamma_{k_n})\big) < \frac{1}{8c}.$$

It follows that

$$\int_0^\infty \mathbf{P}^2(\vartheta(\|S_{k_n}\|) > t)\,dt \leq \int_0^{A\vartheta(\gamma_{k_n})} 1dt + \int_{A\vartheta(\gamma_{k_n})}^\infty \frac{1}{8c}\mathbf{P}\big(\vartheta(\|S_{k_n}\|) > t\big)dt$$
$$\leq A\vartheta(\gamma_{k_n}) + \frac{1}{8c}\mathbf{E}\,\vartheta(\|S_{k_n}\|). \tag{4.4}$$

Thus, by (4.3) and (4.4), we get Lemma 4.3. □

The following lemma is a generalization of Lemma 2.1 of Gut [8] and Lemma 2.7 (b) of Fazekas [6].

Lemma 4.4. *Let $\{X_{nk}, n \in \mathbf{N}, k = 1, \ldots, k_n\}$ be an array of B-valued r.v.'s which is w.m.d. by the r.v. X. If $\vartheta \in \Phi_0$ then*

$$\frac{1}{k_n}\sum_{k=1}^{k_n}\mathbf{E}\,\vartheta(\|X_{nk}\|) \leq (1 \vee \gamma)\mathbf{E}\,\vartheta(|X|).$$

Proof. Using $\vartheta \in \Phi_0$ and (2.4), we have for all $x \geq 0$

$$\frac{1}{k_n}\sum_{k=1}^{k_n}\mathbf{P}\big(\vartheta(\|X_{nk}\|) > \vartheta(x)\big) \leq \frac{1}{k_n}\sum_{k=1}^{k_n}\mathbf{P}(\|X_{nk}\| > x)$$
$$\leq \gamma\mathbf{P}(|X| > x) \leq \gamma\mathbf{P}\big(\vartheta(|X|) \geq \vartheta(x)\big),$$

hence

$$\frac{1}{k_n}\sum_{k=1}^{k_n}\mathbf{P}\big(\vartheta(\|X_{nk}\|) > t\big) \leq (1 \vee \gamma)\mathbf{P}\big(\vartheta(|X|) \geq t\big) \tag{4.5}$$

for all $t \in R_\vartheta$. Assume that $t \in (\vartheta(0), \sup R_\vartheta) \cap \overline{R_\vartheta}$. Then there exists $a \geq 0$, that $\lim_{x\to a-0}\vartheta(x) < t < \lim_{x\to a+0}\vartheta(x)$. (Let $\lim_{x\to 0-0}\vartheta(x) = \vartheta(0)$.) If $\vartheta(a) > t$, then $\{\vartheta(\|X_{nk}\|) > t\} = \{\|X_{nk}\| \geq a\}$. So, by (2.5),

$$\frac{1}{k_n}\sum_{k=1}^{k_n}\mathbf{P}\big(\vartheta(\|X_{nk}\|) > t\big) = \frac{1}{k_n}\sum_{k=1}^{k_n}\mathbf{P}(\|X_{nk}\| \geq a)$$
$$\leq \gamma\mathbf{P}(|X| \geq a) \leq \gamma\mathbf{P}\big(\vartheta(|X|) \geq \vartheta(a)\big) \leq \gamma\mathbf{P}\big(\vartheta(|X|) \geq t\big).$$

Thus (4.5) is true in this case as well. If $\vartheta(a) < t$, then $\{\vartheta(\|X_{nk}\|) > t\} = \{\|X_{nk}\| > a\}$. So, by (2.4),

$$\frac{1}{k_n}\sum_{k=1}^{k_n}\mathbf{P}\big(\vartheta(\|X_{nk}\|) > t\big) = \frac{1}{k_n}\sum_{k=1}^{k_n}\mathbf{P}(\|X_{nk}\| > a) \le \gamma\mathbf{P}(|X| > a) \le \gamma\mathbf{P}\big(\vartheta(|X|) \ge t\big).$$

Thus (4.5) is true in this case as well. Finally, (4.5) is obvious if $0 \le t \le \vartheta(0)$ or $t \ge \sup R_\vartheta$. Thus we have (4.5) for all $t \ge 0$, which implies Lemma 4.4. □

Proof of Theorem 3.1. First assume that X_{nk} are symmetric. Let $\varepsilon > 0$. Using Lemma 4.2 and (2.4), we get

$$\mathbf{P}\left(\|S_n\| > \varepsilon 3^j\alpha(n)\right) \le A_j\gamma n\mathbf{P}\big(|X| > \varepsilon\alpha(n)\big) + B_j\mathbf{P}^{2^j}\left(\|S_n\| > \varepsilon\alpha(n)\right). \tag{4.6}$$

To estimate of second term of (4.6) we can apply $\vartheta \in \Phi_0$, $\vartheta \sim \Delta_2$, Chebyshev's inequality, Lemma 4.3 and Lemma 4.4. Thus there exist $\varepsilon', \gamma', a, b > 0$ such that for all $n \ge n_0$

$$\begin{aligned}\mathbf{P}\left(\frac{1}{\varepsilon}\|S_n\| > \alpha(n)\right) &\le \mathbf{P}\Big(\varepsilon'\vartheta(\|S_n\|) \ge \vartheta(\alpha(n))\Big)\\ &\le \varepsilon'\frac{\mathbf{E}\,\vartheta(\|S_n\|)}{\vartheta(\alpha(n))} \le \frac{\varepsilon'}{\vartheta(\alpha(n))}\Big(a\gamma' n\mathbf{E}\,\vartheta(|X|) + b\vartheta(\gamma_n)\Big).\end{aligned} \tag{4.7}$$

In formula (4.7) we can choose b such that

$$b > \frac{a}{r}\gamma'\mathbf{E}\,\vartheta(|X|), \tag{4.8}$$

where r is from (3.2). Now, (4.6), (4.7) and (4.8) imply that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\mathbf{P}\Big(\|S_n\| > \varepsilon 3^j\alpha(n)\Big) &\le A_j\gamma\sum_{n=1}^{\infty}\mu(n)\mathbf{P}\big(|X| > \varepsilon\alpha(n)\big) + \text{const.}+\\ &+ \text{const.}\sum_{n=n_0+1}^{\infty}\frac{\mu(n)}{n}\left(\frac{rn + \vartheta(\gamma_n)}{\vartheta(\alpha(n))}\right)^{2^j}.\end{aligned} \tag{4.9}$$

Since $\varphi \sim \Delta_2$ there exists $k > 0$ such that $\mathbf{E}\,\varphi\left(|X|/\varepsilon\right) \le k\mathbf{E}\,\varphi(|X|) < \infty$. Thus, by Lemma 4.1 and (3.1), there exists $n_1 \in \mathbf{N}$ such that

$$\infty > \sum_{n=1}^{\infty}\beta(n-1)\mathbf{P}\left(\frac{|X|}{\varepsilon} > \alpha(n)\right) \ge \text{const.}\sum_{n=n_1}^{\infty}\mu(n)\mathbf{P}\big(|X| > \varepsilon\alpha(n)\big). \tag{4.10}$$

Then (4.9), (4.10) and (3.2) imply (3.3).

In the general case let X'_{nk} be an independent copy of X_{nk} for any $n \in \mathbf{N}$ and $k = 1, \ldots, n$. Let $X^*_{nk} = X_{nk} - X'_{nk}$, $S'_n = \sum_{k=1}^n X'_{nk}$ and $S^*_n = \sum_{k=1}^n X^*_{nk} = S_n - S'_n$. We shall prove that conditions of Theorem 3.1 hold for X^*_{nk}.

Using (2.3) and (2.4), we get

$$\frac{1}{n}\sum_{k=1}^{n}\mathbf{P}\left(\|X_{nk}^{*}\|>t\right)\leq\frac{2}{n}\sum_{k=1}^{n}\mathbf{P}\left(\|X_{nk}\|>\frac{t}{2}\right)\leq 2\gamma\mathbf{P}(|2X|>t)\quad\text{for all}\quad t\geq 0$$

so $\{X_{nk}^{*}: n\in\mathbf{N}, k=1,\dots,n\}$ is w.m.d. by $2X$. Moreover, it follows from $\varphi,\vartheta\sim\Delta_2$ that $\mathbf{E}\,\varphi(|2X|)<\infty$ and $\mathbf{E}\,\vartheta(|2X|)<\infty$.

Since $\{\|S_n\|/\gamma_n, n\in\mathbf{N}\}$ is bounded in probability, using (2.3), for every $h>0$ there exists $q>0$ such that for all $n\in\mathbf{N}$

$$2h>2\mathbf{P}(\|S_n\|>q\gamma_n)\geq\mathbf{P}(\|S_n^{*}\|>2q\gamma_n).$$

Thus $\{\|S_n^{*}\|/\gamma_n, n\in\mathbf{N}\}$ is bounded in probability. Therefore the symmetric case implies

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\mathbf{P}\big(\|S_n^{*}\|>\varepsilon\alpha(n)\big)<\infty\quad\text{for all}\quad\varepsilon>0. \tag{4.11}$$

$\{\|S_n'\|/\gamma_n, n\in\mathbf{N}\}$ is bounded in probability as well, so there exists $q'>0$ such that

$$\mathbf{P}(\|S_n'\|<q'\gamma_n)>\frac{1}{2}. \tag{4.12}$$

Finally, (2.3), $\alpha(n)/\gamma_n\to\infty$ and (4.12) imply that for $n\in\mathbf{N}$ large enough

$$\begin{aligned}\mathbf{P}\big(\|S_n^{*}\|>\varepsilon\alpha(n)\big)&\geq\mathbf{P}\big(\|S_n'\|<\varepsilon\alpha(n)\big)\mathbf{P}\big(\|S_n\|>2\varepsilon\alpha(n)\big)\\&\geq\mathbf{P}\big(\|S_n'\|<q'\gamma_n\big)\mathbf{P}\big(\|S_n\|>2\varepsilon\alpha(n)\big)\geq\frac{1}{2}\mathbf{P}\big(\|S_n\|>2\varepsilon\alpha(n)\big).\end{aligned}$$

This fact and (4.11) imply (3.3). □

Proof of Corollary 3.2. In Theorem 3.1 put $\alpha(x)=x^{r/t}$, $\varphi(x)=M^{r/2}(\psi(x^{t/r}))$, $\vartheta(x)=x^{s}$ and $\gamma_n=n^{1/s}$. Then

$$\beta(k_n-1)=\varphi(\alpha(k_n))-\varphi(\alpha(k_n-1))=M^{r/2}(n)-M^{r/2}(n-1) \tag{4.13}$$

and $\beta(m-1)=0$ if $k_n<m<k_{n+1}$ for all $n\in\mathbf{N}$. Using relation $M^{v}(n)-M^{v}(n-1)=\int_{M(n-1)}^{M(n)}vt^{v-1}dt$ it is easy to see that

$$vk_nM^{v-1}(n-1)\leq M^{v}(n)-M^{v}(n-1)\leq vk_n^{2v-1}\quad\text{for all}\quad v\geq 1,\ n\in\mathbf{N} \tag{4.14}$$

and

$$vk_nM^{v-1}(n)\leq M^{v}(n)-M^{v}(n-1)\leq vk_n\quad\text{for all}\quad 0<v\leq 1,\ n\in\mathbf{N}. \tag{4.15}$$

Let $j \in \mathbf{N}$ such that $2^j > t((r-1) \vee 1)/(rs-t)$. Then, using (4.13), (4.14) and (4.15), we have

$$\sum_{n=1}^{\infty} \frac{\beta(k_n-1)}{k_n} \left(\frac{rk_n + \vartheta(\gamma_{k_n})}{\vartheta(\alpha(k_n))} \right)^{2^j} \leq \begin{cases} \text{const.} \sum_{n=1}^{\infty} n^{r-2-(sr/t-1)2^j} < \infty, & \text{if } r > 2, \\ \text{const.} \sum_{n=1}^{\infty} n^{-(sr/t-1)2^j} < \infty, & \text{if } 0 < r \leq 2. \end{cases}$$

It is easy to see that the other conditions of Theorem 3.1 satisfy as well. Thus

$$\sum_{n=1}^{\infty} \frac{\beta(k_n-1)}{k_n} \mathbf{P}\left(\|S_{k_n}\| > \varepsilon k_n^{r/t} \right) < \infty \quad \text{for all} \quad \varepsilon > 0.$$

Furthermore, by (4.13), (4.14) and (4.15), we have $\beta(k_n-1)/k_n \geq \text{const.}(M(n))^{r/2-1}$ which imply the statement. □

Proof of Corollary 3.3. In Theorem 3.1 put $\alpha(x) = x^{1/t}$, $\varphi(x) = M_r(\psi(x^t))$, $\vartheta(x) = x^s$ and $\gamma_n = n^{1/s}$. Then

$$\beta(k_n-1) = \varphi(\alpha(k_n)) - \varphi(\alpha(k_n-1)) = M_r(n) - M_r(n-1) = k_n^{r-1} \tag{4.16}$$

and $\beta(m-1) = 0$ if $k_n < m < k_{n+1}$ for all $n \in \mathbf{N}$. Let $j \in \mathbf{N}$ such that $2^j > t(r-1)/(s-t)$. Then, using (4.16), we have

$$\sum_{n=1}^{\infty} \frac{\beta(k_n-1)}{k_n} \left(\frac{rk_n + \vartheta(\gamma_{k_n})}{\vartheta(\alpha(k_n))} \right)^{2^j} \leq \text{const.} \sum_{n=1}^{\infty} n^{r-2-(s/t-1)2^j} < \infty.$$

It is easy to see that the others conditions of Theorem 3.1 satisfy as well. Thus

$$\sum_{n=1}^{\infty} \frac{\beta(k_n-1)}{k_n} \mathbf{P}\left(\|S_{k_n}\| > \varepsilon k_n^{1/t} \right) < \infty \quad \text{for all} \quad \varepsilon > 0.$$

Finally, by (4.16), we have $\beta(k_n-1)/k_n = k_n^{r-2}$ which imply the statement. □

5. Special cases of the main theorem

If B has an appropriate geometric property, then a moment condition can imply the boundedness of $\{\|S_{k_n}\| / \gamma_{k_n}, n \in \mathbf{N}\}$.

Definition 5.1. For an Orlicz function φ the *Orlicz space* $l_\varphi(B)$ consists of those B-valued sequences $\{u_n, n \in \mathbf{N}\}$ for which

$$\sum_{n=1}^{\infty} \varphi\left(\frac{\|u_n\|}{a} \right) < \infty \quad \text{for some} \quad a > 0.$$

Let $\varepsilon_1, \varepsilon_2, \dots$ be independent r.v.'s with $\mathbf{P}(\varepsilon_n = 1) = \mathbf{P}(\varepsilon_n = -1) = 1/2$ for all $n \in \mathbf{N}$. B is said to be of *type* φ, if $\sum_{n=1}^{\infty} \varepsilon_n u_n$ converges in probability for all $\{u_n, n \in \mathbf{N}\} \in l_\varphi(B)$.

When $\varphi(x) = x^p$ we obtain the well-known notion of type p.

Definition 5.2. B is said to be of (*Rademacher*) *type* p $(0 < p \le 2)$, if $\sum_{n=1}^{\infty} \varepsilon_n u_n$ converges almost surely whenever $\{u_n, n \in \mathbf{N}\} \subseteq B$ with $\sum_{n=1}^{\infty} \|u_n\|^p < \infty$.

Remark 5.3. Let φ be an Orlicz function and $\varphi \sim \Delta_2^0$. B is of type φ iff there exists a constant $c > 0$ such that

$$\mathbf{E}\left\|\sum_{k=1}^{n} X_k\right\| \le c\mathbf{E} \inf_{y>0} \left\{\frac{1}{y}\left(1 + \sum_{k=1}^{n} \varphi(y\|X_k\|)\right)\right\} \tag{5.1}$$

for all $n \in \mathbf{N}$ and every independent B-valued r.v.' $X_1, \dots, X_n$ with $\mathbf{E}X_k = \mathbf{0}$, $k = 1, \dots, n$. (For the proof see Fazekas [5].)

Remark 5.4. There is no Banach space of type p for $p > 2$. Every Banach space is of type p for $0 < p \le 1$. If B is of type p then B is of type p' for all $0 < p' \le p$. It is known that B is of type p iff there exists a $c > 0$ such that

$$\mathbf{E}\left\|\sum_{i=1}^{n} X_i\right\|^p \le c\sum_{i=1}^{n} \mathbf{E}\|X_i\|^p \tag{5.2}$$

for every independent B-valued r.v.'s $X_1, \dots, X_n$ with $\mathbf{E}\|X_i\|^p < \infty$ (and $\mathbf{E}X_i = \mathbf{0}$ if $p \ge 1$), $i = 1, \dots, n$.

The following remarks show that in Theorem 3.1 we can write moment conditions instead of the boundedness of $\{\|S_{k_n}\|/\gamma_{k_n}, n \in \mathbf{N}\}$ if B is of type φ (type p).

Remark 5.5. Let φ be a submultiplicative Orlicz function and let B be a space of type φ. Let $\{X_{nk}, n \in \mathbf{N}, k = 1, \dots, k_n\}$ be an array of rowwise independent B-valued r.v.'s which is w.m.d. by the r.v. X. Assume that $\mathbf{E}X_{nk} = \mathbf{0}$, $k = 1, \dots, k_n$ and $\mathbf{E}\,\varphi(|X|) < \infty$. If sequence $\{k_n \varphi(1/\gamma_{k_n}), n \in \mathbf{N}\}$ is bounded for some sequence $\{\gamma_n, n \in \mathbf{N}\}$ of positive real numbers, then $\{\|S_{k_n}\|/\gamma_{k_n}, n \in \mathbf{N}\}$ is bounded in probability.

Proof. The statement is a consequence of (5.1) and Lemma 4.4:

$$\begin{aligned}
\mathbf{E}\frac{\|S_{k_n}\|}{\gamma_{k_n}} &\le \frac{c}{\gamma_{k_n}}\mathbf{E}\inf_{y>0}\left\{\frac{1}{y}\left(1 + \sum_{k=1}^{k_n}\varphi(y\|X_{nk}\|)\right)\right\} \\
&\le c\mathbf{E}\left(1 + \sum_{k=1}^{k_n}\varphi\left(\frac{\|X_{nk}\|}{\gamma_{k_n}}\right)\right) \le c\left(1 + \varphi\left(\frac{1}{\gamma_{k_n}}\right)(1 \vee \gamma)k_n \mathbf{E}\,\varphi(|X|)\right). \qquad \square
\end{aligned}$$

Remark 5.6. Let B be of type p for some $0 < p \le 2$ and $\{X_{nk}, n \in \mathbf{N}, k = 1, \dots, k_n\}$ be an array of rowwise independent B-valued r.v.'s which is w.m.d. by the r.v. X. Assume that $\mathbf{E}X_{nk} = \mathbf{0}$ $(k = 1, \ldots, k_n)$ in case $p \geq 1$. If $\mathbf{E}|X|^p < \infty$ then $\{\|S_{k_n}\| / k_n^{1/p}, n \in \mathbf{N}\}$ is bounded in probability.

Proof. Using (5.2) and Lemma 4.4, we have

$$\mathbf{E}\|S_{k_n}\|^p \leq c \sum_{k=1}^{k_n} \mathbf{E}\|X_{nk}\|^p \leq c(1 \vee \gamma) k_n \mathbf{E}|X|^p .$$

So $\{\|S_{k_n}\|^p / k_n, n \in \mathbf{N}\}$ is bounded in probability. □

The following corollary is a version of Corollary 4.2 of Hu et al. [12].

Corollary 5.7. *Let $r \in \mathbf{R}$, $0 < p \leq 2$, $0 < t < p$, $M_r \circ \psi \sim \Delta_2$ and B be of type p. Let $\{X_{nk}, n \in \mathbf{N}, k = 1, \ldots, k_n\}$ be an array of rowwise independent B-valued r.v.'s which is w.m.d. by the r.v. X. If $\mathbf{E}X_{nk} = \mathbf{0}$ for all $n \in \mathbf{N}$, $k = 1, \ldots, k_n$,*

$$\mathbf{E}M_r\left(\psi\left(|X|^t\right)\right) < \infty \quad \textit{and} \quad \mathbf{E}|X|^p < \infty,$$

then

$$\sum_{n=1}^{\infty} k_n^{r-2} \mathbf{P}\left(\|S_{k_n}\| > \varepsilon k_n^{1/t}\right) < \infty \quad \textit{for all} \quad \varepsilon > 0.$$

Proof. It follows from Remark 5.6 that $\{\|S_{k_n}\| / k_n^{1/p}, n \in \mathbf{N}\}$ is bounded in probability. Hence conditions of Corollary 3.3 are satisfied. □

The following three theorems are due to Fazekas [6]. We shall prove, that they are special cases of Theorem 3.1.

Theorem 5.8. (Theorem 3.1 of Fazekas [6].) *Let $0 < p \leq 2$, $s \geq p$, $rp > s$ and let B be of type p. Let $\{X_{nk}, n \in \mathbf{N}, k = 1, \ldots, n\}$ be an array of rowwise independent B-valued r.v.'s which is w.m.d. by the r.v. X. Assume that $\mathbf{E}X_{nk} = \mathbf{0}$ $(k = 1, \ldots, n)$ in case $p \geq 1$. If $\mathbf{E}|X|^s < \infty$, then*

$$\sum_{n=1}^{\infty} n^{r-2} \mathbf{P}\left(\|S_n\| > \varepsilon n^{r/s}\right) < \infty \quad \textit{for all} \quad \varepsilon > 0.$$

Proof. In Theorem 3.1 put $\alpha(x) = x^{r/s}$, $\varphi(x) = \vartheta(x) = x^s$ and $\gamma_n = n^{1/p}$. Then

$$\beta(n) = \varphi(\alpha(n+1)) - \varphi(\alpha(n)) = (n+1)^r - n^r.$$

Let $j \in \mathbf{N}$ such that $2^j > rp/(rp - s)$. Then

$$\sum_{n=1}^{\infty} \frac{\beta(n)}{n} \left(\frac{rn + \vartheta(\gamma_n)}{\vartheta(\alpha(n))}\right)^{2^j} \leq \text{const.} \sum_{n=1}^{\infty} n^{r-1-(r-s/p)2^j} < \infty,$$

hence (3.2) holds. Since $\beta(n-1)/\beta(n) \to 1$ thus (3.1) is satisfied. It follows from $\mathbf{E}\,|X|^s < \infty$ that $\mathbf{E}\,|X|^p < \infty$. Hence by Remark 5.6, $\{\|S_n\|/n^{1/p}, n \in \mathbf{N}\}$ is bounded in probability. It is easy to see that the other conditions of Theorem 3.1 hold true as well. Thus

$$\sum_{n=1}^{\infty} \frac{(n+1)^r - n^r}{n} \mathbf{P}\big(\|S_n\| > \varepsilon n^{r/s}\big) < \infty \quad \text{for all} \quad \varepsilon > 0.$$

This implies the statement, because $((n+1)^r - n^r)/n \geq \text{const.} n^{r-2}$ for all $n \in \mathbf{N}$. □

Theorem 5.9. (Theorem 3.5 of Fazekas [6] and Theorem 3.3 of Jain [13].) *Let $\{X_{nk}, n \in \mathbf{N}, k = 1, \ldots, n\}$ be an array of rowwise independent B-valued r.v.'s which is w.m.d. by the r.v. X. Let $\alpha, \varphi \in \Phi_0$, which are strictly increasing, $R_\alpha = R_\varphi = [0,\infty)$ and $\varphi \sim \Delta_2$. Let $\beta(n) = \varphi(\alpha(n+1)) - \varphi(\alpha(n))$ such that for some $c_1, c_2 > 0$*

$$c_1 \leq c_2 \beta(n+1) \leq \beta(n) \quad \textit{for all} \quad n \in \mathbf{N}.$$

Let $\mathbf{E}\,\varphi(|X|) < \infty$. Assume that there exists a sequence $\{\gamma_n, n \in \mathbf{N}\}$ of positive real numbers such that $\{\|S_n\|/\gamma_n, n \in \mathbf{N}\}$ is bounded in probability, moreover there exists $\delta > 0$ such that

$$\frac{n \vee \varphi(\gamma_n)}{\varphi(\alpha(n))} = O\big((\log n)^{-\delta} \wedge (\beta(n))^{-\delta}\big). \tag{5.3}$$

Then

$$\sum_{n=1}^{\infty} \frac{\beta(n)}{n} \mathbf{P}\big(\|S_n\| > \varepsilon \alpha(n)\big) < \infty \quad \text{for all} \quad \varepsilon > 0. \tag{5.4}$$

Proof. In Theorem 3.1 put $\vartheta = \varphi$ and choose $j \in \mathbf{N}$ such that $2^j > 2/\delta$. Then, using (5.3) and $1/\beta(n) \leq 1/c_1$, we get for some $m_0 \in \mathbf{N}$ that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\beta(n)}{n} \left(\frac{rn + \vartheta(\gamma_n)}{\vartheta(\alpha(n))}\right)^{2^j} &\leq \text{const.} + \text{const.} \sum_{n=m_0}^{\infty} \frac{\beta(n)}{n} \left(\frac{r+1}{(\beta(n)\log n)^{\delta/2}}\right)^{2^j} \\ &\leq \text{const.} + \text{const.} \sum_{n=m_0}^{\infty} n^{-1} (\log n)^{-\delta 2^{j-1}} < \infty.\end{aligned}$$

It follows from (5.3) that $\text{const.}(\log n)^\delta \leq \varphi(\alpha(n))/\varphi(\gamma_n)$ for $n \in \mathbf{N}$ large enough, hence $\varphi(\alpha(n))/\varphi(\gamma_n) \to \infty$. This fact and $\varphi \sim \Delta_2$ imply that $\alpha(n)/\gamma_n \to \infty$. Consequently, Theorem 3.1 implies (5.4). □

Theorem 5.10. (Theorem 6.2 of Fazekas [6].) *Let $\{X_{nk}, n \in \mathbf{N}, k = 1, \ldots, k_n\}$ be an array of rowwise independent B-valued r.v.'s which is w.m.d. by the r.v. X. Let $0 < p \leq 2$, $r \geq 1$, $t > 0$ and $s \geq p$. Suppose that $r > t/p$ if $s > 1$ while $r > t/s$ if $s \leq 1$. In case $r > 2$ assume that*

$$\limsup_{n\to\infty} \frac{k_n}{M(n-1)} < \infty. \tag{5.5}$$

Let $M \circ \psi \sim \Delta_2$ and B be of type p. Assume that $\mathbf{E}X_{nk} = \mathbf{0}$ $(k = 1, \dots, k_n)$ in case $p \geq 1$. If

$$\mathbf{E}M^{r/2}\Big(\psi\big(|X|^{t/r}\big)\Big) < \infty \quad \textit{and} \quad \mathbf{E}\,|X|^s < \infty,$$

then

$$\sum_{n=1}^{\infty} \big(M(n)\big)^{r/2-1} \mathbf{P}\left(\|S_{k_n}\| > \varepsilon k_n^{r/t}\right) < \infty \quad \textit{for all} \quad \varepsilon > 0.$$

Proof. Let $q = p$ if $s > 1$ while $q = s$ if $s \leq 1$. Then $rq > t$, B is of type q and $\mathbf{E}\,|X|^q < \infty$. Hence, using Remark 5.6, we get that $\{\|S_{k_n}\| / k_n^{1/q}, n \in \mathbf{N}\}$ is bounded in probability. On the other hand $\limsup_{n\to\infty} k_n/M(n-1) < \infty$ implies the boundedness of $\{M(n)/M(n-1), n \in \mathbf{N}\}$. So every condition of Corollary 3.2 are satisfied. □

In the following we shall consider Corollary 3.2 and Corollary 3.3 for particular sequences k_n. To this end we have to know whether $M_r \circ \psi \sim \Delta_2$ holds true.

Remark 5.11. If any of the conditions (a), (b), (c) and (d) holds, then $M_r \circ \psi \sim \Delta_2$, where

(a) $k_{n+1}^{r-1}/M_r(n)$ and $M_r(\psi(2k_n))/M_r(n)$ are bounded,

(b) $M_r(\psi(2k_{n+1}))/M_r(n)$ is bounded,

(c) $r \in \mathbf{N}$ and $k_n = n^d$ ($d \in \mathbf{N}$ is fixed),

(d) $r \geq 1$ and $k_n = q^n$ ($q = 2, 3, \dots$ is fixed).

Proof. (a) Using $M_r(n+1)/M_r(n) = 1 + k_{n+1}^{r-1}/M_r(n)$, we get, there exists $K > 1$ such that $M_r(n+1) \leq KM_r(n)$ for all $n \in \mathbf{N}$. Assume that $t \geq k_1$ and let $m = \psi(t)$. Then there exists $L \in \mathbf{R}$ such that

$$M_r(\psi(2t)) \leq M_r(\psi(2k_{m+1})) \leq LM_r(m+1) \leq KLM_r(m) = KLM_r(\psi(t)).$$

If $0 < t < k_1$, then $\psi(t) = 0$, so

$$M_r(\psi(2t)) \leq M_r(\psi(2k_1)) \leq LM_r(1) \leq KLM_r(0) = KLM_r(\psi(t)).$$

(b) $1+k_{n+1}^{r-1}/M_r(n)$ and $M_r(\psi(2k_n))/M_r(n)$ are not greater than $M_r(\psi(2k_{n+1}))/M_r(n)$, so (a) implies (b).

(c) $M_r(\psi(2k_{n+1}))/M_r(n) \leq M_r(2n+2)/M_r(n) \to 2^{d(r-1)+1}$, consequently (b) implies the statement.

(d) $M_r(\psi(2k_{n+1}))/M_r(n) \leq M_r(n+2)/M_r(n) \to q^{2(r-1)}$, consequently (b) implies the statement. □

In the special case $k_n = n^d$ ($d \in \mathbf{N}$ is fixed) from Corollary 3.2 we obtain the following statement.

Remark 5.12. Let $\{X_{nk}, n \in \mathbf{N}, k = 1, \ldots, n^d\}$ be an array of rowwise independent B-valued r.v.'s which is w.m.d. by the r.v. X. Let $r, s, t > 0$, $rs > t$ and $w = t(d+1)/(2d)$. If $\{\|S_{n^d}\| / n^{d/s}, n \in \mathbf{N}\}$ is bounded in probability, $\mathbf{E}|X|^s < \infty$ and $\mathbf{E}|X|^w < \infty$, then

$$\sum_{n=1}^{\infty} n^{(d+1)(r/2-1)} \mathbf{P}\left(\|S_{n^d}\| > \varepsilon n^{dr/t}\right) < \infty \quad \text{for all} \quad \varepsilon > 0.$$

Proof. Let $Y = M^{r/2}(\psi(|X|^{t/r}))$. Then

$$\begin{aligned} Y &= Y\mathbf{I}(|X| < 1) + Y\mathbf{I}(|X| \geq 1) \\ &\leq \mathbf{I}(|X| < 1) + (\text{const.}|X|^{2w/r})^{r/2}\mathbf{I}(|X| \geq 1) \leq 1 + \text{const.}|X|^w. \end{aligned}$$

So $\mathbf{E}Y < \infty$. It is easy to see that the other conditions of Corollary 3.2 hold true as well. On the other hand $M(n) \geq \text{const.}n^{d+1}$. □

In the special case $k_n = q^n$ (where $q = 2, 3, \ldots$ is fixed) from Corollary 3.3 we get the following statement.

Remark 5.13. Let $\{X_{nk}, n \in \mathbf{N}, k = 1, \ldots, q^n\}$ be an array of rowwise independent B-valued r.v.'s which is w.m.d. by the r.v. X. Let $0 < t < s$, $w > -1$. If $\{\|S_{q^n}\| / q^{n/s}, n \in \mathbf{N}\}$ is bounded in probability, $\mathbf{E}|X|^s < \infty$ and $\mathbf{E}|X|^{t(w+1)} < \infty$ then

$$\sum_{n=1}^{\infty} q^{nw} \mathbf{P}\left(\|S_{q^n}\| > \varepsilon q^{n/t}\right) < \infty \quad \text{for all} \quad \varepsilon > 0.$$

Proof. Let $r = w + 2$ and $Y = M_r(\psi(|X|^t))$. Then

$$\begin{aligned} Y &= Y\mathbf{I}(|X|^t < q) + Y\mathbf{I}(|X|^t \geq q) \\ &\leq q^{r-1}\mathbf{I}(|X|^t < q) + \text{const.}|X|^{t(r-1)}\mathbf{I}(|X|^t \geq q) \leq q^{r-1} + \text{const.}|X|^{t(w+1)}. \end{aligned}$$

So $\mathbf{E}Y < \infty$. The other conditions of Corollary 3.3 hold true as well. □

Acknowledgement. The author would like to thank István Fazekas for several helpful discussions and his careful attention my paper.

References

[1] S. E. Ahmed, R. G. Antonini, A. Volodin, On the rate of complete convergence for weighted sums of arrays of Banach space valued random elements with application to moving average processes, Statist. Probab. Letters, 58 (2002), 185–194.

[2] L. E. Baum and M. Katz, Convergence rates in the law of large numbers, Trans. Amer. Math. Soc. 120 (1965), 108–123.

[3] P. Erdős, On a theorem of Hsu and Robbins, Ann. Math. Statist. 20 (1949), 286–291.
[4] P. Erdős, Remark on my paper "On a theorem of Hsu and Robbins", Ann. Math. Statist. 21 (1950), 138.
[5] I. Fazekas, A strong law of large numbers in Banach spaces of type Φ, In Mathematical Statistics and Probability Theory, Vol. A, 89–97, Reidel, Dordrecht, 1987.
[6] I. Fazekas, Convergence rates in the law of large numbers for arrays, Publ. Math. Debrecen, 41/1-2 (1992), 53–71.
[7] A. Gut, On complete convergence in the law of large numbers for subsequences, Ann. Probab. 13 (1985), 1286–1291.
[8] A. Gut, Complete convergence for arrays, Periodica Math. Hungar. 25 (1), (1992), 51–75.
[9] J. Hoffmann-Jørgensen, Sums of independent Banach space valued random variables, Studia Math. LII (1974), 159–186.
[10] P. L. Hsu and H. Robbins, Complete convergence and the law of large numbers, Proc. Nat. Acad. Sci. USA 33 (1947), 25–31.
[11] T.-C. Hu, F. Móricz and R. L. Taylor, Strong laws of large numbers for arrays of rowwise independent random variables, Acta Math. Acad. Sci. Hungar. 54 (1989), 153–162.
[12] T.-C. Hu, A. Rosalsky, D. Szynal and A. I. Volodin, On complete convergence for arrays of rowwise independent random elements in Banach spaces, Stoch. Anal. Appl., 17 (1999) no 6., 963–992.
[13] N. C. Jain, Tail probabilities for sums of independent Banach space valued random variables, Z. Wahrscheinlichkeitstheorie verw. Gebiete 33 (1975), 155–166.
[14] F. L. Spitzer, A combinatorial lemma and its application, Trans. Amer. Math. Soc. 82 (1956), 323–339.
[15] W. A. Woyczyński, On Marcinkiewicz–Zygmund laws of large numbers in Banach spaces and related rates of convergence, Probab. Math. Statist. 1 (1980), 117—131.

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