ASYMPTOTIC RESULTS
IN PROBABILITY THEORY

Theses of PhD dissertation

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Contents

1. Strong laws of large numbers for pairwise independent random variables with multidimensional indices

Let $\mathbb{N}^d$ be the positive integer d-dimensional lattice points, where d is a positive integer. The following notation will be used: $|\mathbf{n}| = n_1 \cdots n_d$, where $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{N}^d$, $\sum_{\mathbf{n}} = \sum_{\mathbf{n}\in\mathbb{N}^d}$, $\log^+ x = \max\{1, \log x\}$, if $x > 0$ and $\log^+ x = 1$, if $x \le 0$. For $\mathbf{n}, \mathbf{m} \in \mathbb{N}^d$, $\mathbf{n} \le \mathbf{m}$ is defined coordinatewise.

We shall assume that random variables (r.v.'s) $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ are defined on the same probability space. Let $S_{\mathbf{n}} = \sum_{\mathbf{k}\le\mathbf{n}} X_{\mathbf{k}}$.

Definition 1.1. (Gut (1992) [8].) It is said that the sequence $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ is *weakly mean dominated* by the r.v. X, if for some $c > 0$,

$$\frac{1}{|\mathbf{n}|} \sum_{\mathbf{k}\le\mathbf{n}} \mathrm{P}(|X_{\mathbf{k}}| > x) \le c\mathrm{P}(|X| > x)$$

for all $\mathbf{n} \in \mathbb{N}^d$ and $x \ge 0$.

The following multiindex convergence theorems are extensions of the single index theorems in Kruglov (1994) [11].

1.1. A general almost surely convergence theorem

Fazekas and Tómács (1998) [7] Theorem 3.1.

Theorem 1.1.1. *Let* $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ *be a sequence of nonnegative r.v.'s, and let* $\{b_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ *be a bounded sequence of nonnegative numbers, and* $B_{\mathbf{n}} = \sum_{\mathbf{k}\le\mathbf{n}} b_{\mathbf{k}}$. *If*

$$\sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} \mathrm{P}\left(|S_{\mathbf{n}} - B_{\mathbf{n}}| > \varepsilon |\mathbf{n}|^{1/r}\right) < \infty$$

for every $\varepsilon > 0$, *where* $0 < r \le 1$, *then*

$$\frac{1}{|\mathbf{n}|^{1/r}} (S_{\mathbf{n}} - B_{\mathbf{n}}) \to 0 \quad (|\mathbf{n}|\to\infty) \quad \textit{almost surely } (\text{a.s.}).$$

1.2. Kolmogorov's SLLN

Fazekas and Tómács (1998) [7] Theorem 4.1.

Theorem 1.2.1. *Let* $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ *be a sequence of pairwise independent r.v.'s, which is weakly mean dominated by* X. *Assume that*

$$\mathrm{E}\left(|X|(\log^+ |X|)^{d-1}\right) < \infty.$$

Then

$$\sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} \mathrm{P}\big(|S_{\mathbf{n}} - \mathrm{E}S_{\mathbf{n}}| > \varepsilon|\mathbf{n}|\big) < \infty$$

for any $\varepsilon > 0$, *moreover if* $\sup\{\mathrm{E}|X_{\mathbf{n}}| : \mathbf{n} \in \mathbb{N}^d\} < \infty$, *then*

$$\frac{1}{|\mathbf{n}|}(S_{\mathbf{n}} - \mathrm{E}S_{\mathbf{n}}) \to 0 \quad (|\mathbf{n}|{\to}\infty) \quad \text{a.s.}$$

1.3. The Marcinkiewicz SLLN without assuming independence

Fazekas and Tómács (1998) [7] Theorem 5.1.

1.3.1. tétel. *Let* $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ *be weakly mean dominated by* X *such that* $\mathrm{E}(|X|^r (\log^+ |X|)^{d-1}) < \infty$, *where* $0 < r < 1$. *Then*

$$\sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} \mathrm{P}\left(|S_{\mathbf{n}}| > \varepsilon|\mathbf{n}|^{1/r}\right) < \infty$$

for any $\varepsilon > 0$, *and*

$$\frac{S_{\mathbf{n}}}{|\mathbf{n}|^{1/r}} \to 0 \quad (|\mathbf{n}|{\to}\infty) \quad \text{a.s.}$$

2. A general approach to strong laws of large numbers for fields of random variables

We shall use notation of the previous chapter. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbf{0} := (0,\dots,0) \in \mathbb{N}_0^d$ and $\mathbf{1} = (1,\dots,1) \in \mathbb{N}^d$. Denote $\mathbb{R}$ the set of real numbers. Coordinates of $\mathbf{m}, \mathbf{n}, \ldots \in \mathbb{N}^d$ are denoted by same letters with indices, i.e. $\mathbf{n}$ always means the vector $(n_1,\dots,n_d)$. $\mathbf{n} \to \infty$ is interpreted as $n_i \to \infty$ for all $i \in \{1,\dots,d\}$. Let $|\log \mathbf{n}| := \prod_{i=1}^d \log^+ n_i$. The *difference sequence* of sequence $\{a_{\mathbf{n}} \in \mathbb{R}, \mathbf{n} \in \mathbb{N}_0^d\}$ will be denoted by $\Delta a_{\mathbf{n}}$, i.e. $\sum_{\mathbf{k} \le \mathbf{n}} \Delta a_{\mathbf{k}} = a_{\mathbf{n}}$. We shall say that a sequence $\{a_{\mathbf{n}} \in \mathbb{R}, \mathbf{n} \in \mathbb{N}_0^d\}$ is of *product type*, if there exist $a_n^{(i)} \in \mathbb{R}$, $n \in \mathbb{N}_0$, $i \in \{1,\dots,d\}$, such that $a_{\mathbf{n}} = \prod_{i=1}^d a_{n_i}^{(i)}$ for each $\mathbf{n} \in \mathbb{N}_0^d$. In this case the sequence $\{a_{\mathbf{n}} \in \mathbb{R}, \mathbf{n} \in \mathbb{N}_0^d\}$ is said to be nondecreasing, (resp. unbounded), if $a_n^{(i)}$, $n \in \mathbb{N}_0$ is nondecreasing, (resp. unbounded) for all $i \in \{1,\dots,d\}$.

Definition 2.1. A function $g\colon \mathbb{N}^d \times \mathbb{N}^d \to \mathbb{R}$ is said to be *superadditive*, if

$$g\big(\mathbf{i}, (j_1,\dots,j_{m-1},k,j_{m+1},\dots,j_d)\big) + \\ + g\big((i_1,\dots,i_{m-1},k+1,i_{m+1},\dots,i_d),\mathbf{j}\big) \le g(\mathbf{i},\mathbf{j})$$

for any $\mathbf{i}, \mathbf{j} \in \mathbb{N}^d$, $\mathbf{i} \le \mathbf{j}$, $m = 1, \ldots, d$, $i_m \le k \le j_m$. A sequence of r.v.'s $\{X_\mathbf{n}, \mathbf{n} \in \mathbb{N}^d\}$ is said to have *r-th moment function of superadditive structure*, if for all $\mathbf{i} \le \mathbf{j}$ ($\mathbf{i}, \mathbf{j} \in \mathbb{N}^d$)

$$\mathrm{E}\left|\sum_{\mathbf{i}\le\mathbf{k}\le\mathbf{j}} X_\mathbf{k}\right|^r \le g^\alpha(\mathbf{i}, \mathbf{j}),$$

where $g\colon \mathbb{N}^d \times \mathbb{N}^d \to \mathbb{R}$ is superadditive, $\alpha > 1$ and $r > 0$.

Definition 2.2. Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space. Let $\{X_\mathbf{n}, \mathbf{n} \in \mathbb{N}^d\}$ and $\{\mathcal{F}_\mathbf{n}, \mathbf{n} \in \mathbb{N}^d\}$ be a sequence of r.v.'s and be a sequence of σ-subalgebras of $\mathcal{F}$, respectively. We assume that $\mathcal{F}_\mathbf{m} \subseteq \mathcal{F}_\mathbf{n}$ if $\mathbf{m} \le \mathbf{n}$. If $X_\mathbf{n}$ is measurable with respect to $\mathcal{F}_\mathbf{n}$ for all $\mathbf{n} \in \mathbb{N}^d$, $\mathrm{E}X_\mathbf{1} = 0$ and $\mathrm{E}(X_\mathbf{n} \mid \mathcal{F}_\mathbf{m}) = 0$ if $\mathbf{m} \le \mathbf{n}$, $\mathbf{m} \ne \mathbf{n}$, then we say that $(X_\mathbf{n}, \mathcal{F}_\mathbf{n})$ is a *martingale difference.*

The following convergence theorems are multiindex versions of the single index results in Fazekas and Klesov (2000) [3].

2.1. The basic SLLN

Noszály and Tómács (2000) [12] Proposition 1 and Fazekas, Klesov, Noszály and Tómács (1999) [4] Theorem 3.1.

Theorem 2.1.1. *Let $\mathbf{n} \in \mathbb{N}^d$ be fixed. Let r be a positive real number, $\{a_\mathbf{m}, \mathbf{m} \in \mathbb{N}_0^d\}$ be a nonnegative sequence. Suppose that $\{b_\mathbf{m}, \mathbf{m} \in \mathbb{N}_0^d\}$ is a positive, nondecreasing sequence of product type. If*

$$\mathrm{E}\left(\max_{\mathbf{l}\le\mathbf{m}} |S_\mathbf{l}|^r\right) \le \sum_{\mathbf{l}\le\mathbf{m}} a_\mathbf{l}$$

for all $\mathbf{m} \le \mathbf{n}$, then

$$\mathrm{E}\left(\max_{\mathbf{m}\le\mathbf{n}} \left|\frac{S_\mathbf{m}}{b_\mathbf{m}}\right|^r\right) \le 4^d \sum_{\mathbf{m}\le\mathbf{n}} \frac{a_\mathbf{m}}{b_\mathbf{m}^r}.$$

Noszály and Tómács (2000) [12] Theorem 3 and Fazekas, Klesov, Noszály and Tómács (1999) [4] Theorem 3.2.

Theorem 2.1.2. *Let $\{a_\mathbf{n}, \mathbf{n} \in \mathbb{N}_0^d\}$, $\{b_\mathbf{n}, \mathbf{n} \in \mathbb{N}_0^d\}$ be nonnegative sequences and let $r > 0$. Suppose that $\{b_\mathbf{n}, \mathbf{n} \in \mathbb{N}_0^d\}$ is a positive, nondecreasing, unbounded sequence of product type. If $\sum_\mathbf{n} a_\mathbf{n}/b_\mathbf{n}^r < \infty$ and $\mathrm{E}\left(\max_{\mathbf{m}\le\mathbf{n}} |S_\mathbf{m}|^r\right) \le \sum_{\mathbf{m}\le\mathbf{n}} a_\mathbf{m}$ for all $\mathbf{n} \in \mathbb{N}^d$, then*

$$\frac{S_\mathbf{n}}{b_\mathbf{n}} \to 0 \quad (\mathbf{n} \to \infty) \quad \text{a.s.}$$

2.2. Applications

Logarithmically weighted sums. Noszály and Tómács (2000) [12] Theorem 7 and Fazekas, Klesov, Noszály and Tómács (1999) [4] Theorem 4.2.

Theorem 2.2.1. *Let $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ be a sequence of random variables and suppose that for some $C > 0$, $\beta > 0$*

$$|\mathrm{E}(X_{\mathbf{k}} X_{\mathbf{l}})| \leq C \left(\frac{|\mathbf{k}|}{|\mathbf{l}|} \right)^{\beta} \frac{1}{(\log^{+} |\mathbf{l}|)^{d-1}}$$

for all $\mathbf{k} \leq \mathbf{l}$ $(\mathbf{k}, \mathbf{l} \in \mathbb{N}^d)$. Then

$$\frac{1}{|\log \mathbf{n}|} \sum_{\mathbf{k} \leq \mathbf{n}} \frac{X_{\mathbf{k}}}{|\mathbf{k}|} \to 0 \quad (\mathbf{n} \to \infty) \quad a.s.$$

Sequences with superadditive moment structures. Noszály and Tómács (2000) [12] Proposition 11 and Fazekas, Klesov, Noszály and Tómács (1999) [4] Theorem 4.1.

Theorem 2.2.2. *Let $r > 0$, $\alpha > 1$ and suppose that $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ has r-th moment function of superadditive structure and $\Delta g^{\alpha}(\mathbf{1}, \mathbf{n})$ is nonnegative for any $\mathbf{n} \in \mathbb{N}^d$. Then for arbitrary $q > 0$*

$$\sum_{\mathbf{n}} \frac{g^{\alpha}(\mathbf{1}, \mathbf{n})}{|\mathbf{n}|^{1+r/q}} < \infty$$

implies

$$\frac{S_{\mathbf{n}}}{|\mathbf{n}|^{1/q}} \to 0 \quad (\mathbf{n} \to \infty) \quad a.s.$$

Brunk—Prohorov type theorems. Noszály and Tómács (2000) [12] Proposition 13.

Theorem 2.2.3. *Let $(X_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}})$ be a martingale difference,*

$$\mathrm{E}\Big(\mathrm{E}(X_{\mathbf{l}} \mid \mathcal{F}_{\mathbf{m}}) \Bigm| \mathcal{F}_{\mathbf{n}} \Big) = \mathrm{E}\left(X_{\mathbf{l}} \mid \mathcal{F}_{\min(\mathbf{m}, \mathbf{n})} \right) \tag{2.2.1}$$

for all $\mathbf{l}, \mathbf{m}, \mathbf{n} \in \mathbb{N}^d$. Let $p \geq 1$, $C > 0$ and $r < p + 1$. Suppose that $\sum_{\mathbf{m} \leq \mathbf{n}} \mathrm{E}|X_{\mathbf{m}}|^{2p} \leq C|\mathbf{n}|^{r}$. Then $S_{\mathbf{n}}/|\mathbf{n}| \to 0$ $(\mathbf{n} \to \infty)$ a.s.

Noszály and Tómács (2000) [12] Proposition 14.

Theorem 2.2.4. *Let* $(X_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}})$ *be a martingale difference having property* (2.2.1) *and let* $p \geq 1$*. When* $p > 1$*, suppose that* $\mathrm{E}|X_{\mathbf{n}}|^{2p}$ *is sequence of product type. Then*

$$\sum_{\mathbf{n}} \frac{\mathrm{E}|X_{\mathbf{n}}|^{2p}}{b_{\mathbf{n}}^{2p}} |\mathbf{n}|^{p-1} < \infty$$

implies $S_{\mathbf{n}}/b_{\mathbf{n}} \to 0$ $(\mathbf{n} \to \infty)$ *a.s., provided that* $b_{\mathbf{n}}$ *is a nondecreasing, positive, unbounded sequence of product type and either* $p = 1$ *or* $|\mathbf{n}|^{\delta}/b_{\mathbf{n}}$ *is nonincreasing for some* $\delta > (p-1)/2p$*.*

3. Convergence rates in the law of large numbers for arrays of Banach space valued random elements

Let Φ_0 denote the set of functions $f\colon [0,\infty) \to [0,\infty)$, that are nondecreasing. A function $f \in \Phi_0$ is said to satisfy the Δ_2-condition ($f \sim \Delta_2$) if there exists a constant $c > 0$, such that $f(2t) \leq cf(t)$ for all $t > 0$.

Let B be a real separable Banach space with norm $\|.\|$, and let $\{X_{nk}, n \in \mathbb{N}, k = 1,\ldots,n\}$ be an array of B-valued r.v.'s. It is *rowwise independent*, if $X_{n1},\ldots,X_{nn}$ are independent r.v.'s for all $n \in \mathbb{N}$. Let $S_n = \sum_{k=1}^{n} X_{nk}$.

Definition 3.1. (Gut (1992) [8].) We say that the array $\{X_{nk}, n \in \mathbb{N}, k = 1,\ldots,n\}$ is *weakly mean dominated* by the r.v. X, if for some $\gamma > 0$,

$$\frac{1}{k_n}\sum_{k=1}^{k_n} \mathrm{P}(\|X_{nk}\| > t) \leq \gamma \mathrm{P}(|X| > t) \quad \text{for all} \quad t \geq 0 \quad \text{and} \quad n \in \mathbb{N}.$$

Tómács (2003) [14] Theorem 3.1.

Theorem 3.2. *Let* $\{X_{nk}, n \in \mathbb{N}, k = 1,\ldots,n\}$ *be an array of rowwise independent* B*-valued r.v.'s which is w.m.d. by the r.v.* X*. We assume that there exists a sequence* $\{\gamma_n, n \in \mathbb{N}\}$ *of positive real numbers such that* $\{\|S_n\|/\gamma_n, n \in \mathbb{N}\}$ *is bounded in probability. Let* $\alpha, \vartheta, \varphi \in \Phi_0$*,* α *be not bounded,* $\vartheta, \varphi \sim \Delta_2$*,* $\vartheta \not\equiv 0$ *and*

$$\beta(n) = \varphi(\alpha(n+1)) - \varphi(\alpha(n)), \quad n = 0, 1, 2, \ldots.$$

We assume that

$$\mathrm{E}\,\varphi(|X|) < \infty, \quad \mathrm{E}\,\vartheta(|X|) < \infty \quad \textit{and} \quad \lim_{n\to\infty} \frac{\alpha(n)}{\gamma_n} = \infty.$$

Let

$$\mu(n) = \beta(n-1) \quad \textit{for all} \quad n \in \mathbb{N}$$

or

$$\mu(n) = \beta(n) \quad \textit{for all} \quad n \in \mathbb{N}.$$

In case $\mu(n) = \beta(n)$ *assume that there exists a constant* $c > 0$ *such that*

$$c\beta(n) \leq \beta(n-1)$$

for $n \in \mathbb{N}$ *large enough. Let* $n_0 \in \mathbb{N}$ *be such that* $\vartheta(\alpha(n)) > 0$ *for all* $n \geq n_0$. *If there exist* $j \in \mathbb{N}$ *and* $r > 0$ *such that*

$$\sum_{n=n_0}^{\infty} \frac{\mu(n)}{n} \left(\frac{rn + \vartheta(\gamma_n)}{\vartheta(\alpha(n))} \right)^{2^j} < \infty$$

then

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \mathrm{P}\big(\|S_n\| > \varepsilon\alpha(n)\big) < \infty \quad \textit{for all} \quad \varepsilon > 0.$$

Remark 3.3. The following known Baum—Katz type results are corollaries of the previous theorem.

- Theorem 3.1, Theorem 3.5 and Theorem 6.2 of Fazekas (1992) [2],
- a version of Corollary 4.1 and 4.2 of Hu, Rosalsky, Szynal and Volodin (1999) [9],
- Theorem 3.3 of Jain (1975) [10].

E.g. Theorem 3.1 of Fazekas (1992) [2] is the following.

Theorem 3.4. *Let* $0 < p \leq 2$, $s \geq p$, $rp > s$ *and let* B *be of type* p. *Let* $\{X_{nk}, n \in \mathbb{N}, k = 1, \ldots, n\}$ *be an array of rowwise independent* B*-valued r.v.'s which is weakly mean dominated by the r.v.* X. *Assume that* $\mathrm{E}X_{nk} = \mathbf{0}$ $(k = 1, \ldots, n)$ *in case* $p \geq 1$. *If* $\mathrm{E}\,|X|^s < \infty$, *then*

$$\sum_{n=1}^{\infty} n^{r-2} \mathrm{P}\left(\|S_n\| > \varepsilon n^{r/s}\right) < \infty \quad \textit{for all} \quad \varepsilon > 0.$$

4. Almost sure central limit theorems for m-dependent random fields

Denote $\mathcal{B}$ the σ-algebra of Borel-sets of $\mathbb{R}$. Let δ_x be the *unit mass* at point $x \in \mathbb{R}$, that is $\delta_x \colon \mathcal{B} \to \mathbb{R}$, $\delta_x(B) = 1$ if $x \in B$ and $\delta_x(B) = 0$ if $x \notin B$. $\Rightarrow \mu$

denotes weak convergence to the probability measure μ. Let $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ be a multiindex sequence of r.v.'s, where d is a fixed positive integer. Suppose that $\mathrm{E}X_{\mathbf{n}} = 0$ and $\mathrm{D}^2 X_{\mathbf{n}} < \infty$ for all $\mathbf{n} \in \mathbb{N}^d$. Let $||\mathbf{n}|| = \max\{n_1, \dots, n_d\}$ and $d(V_1, V_2) = \inf\{||\mathbf{n} - \mathbf{m}|| : \mathbf{n} \in V_1, \mathbf{m} \in V_2\}$, where $V_1, V_2 \subset \mathbb{N}^d$. Let $\sigma(V)$ (where $V \subset \mathbb{N}^d$) be the smallest σ-algebra with respect to which $\{X_{\mathbf{n}}, \mathbf{n} \in V\}$ are measurable.

Recall the well-known concept of m-depence.

Definition 4.1. Let $m \in \mathbb{N}$ be fixed. The random field $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ is said to be *m-dependent*, if the σ-algebras $\sigma(V_1)$ and $\sigma(V_2)$ are independent whenever $d(V_1, V_2) > m$, $V_1, V_2 \subset \mathbb{N}^d$.

In the following let $S_{\mathbf{n}} = \sum_{\mathbf{k} \le \mathbf{n}} X_{\mathbf{k}}$, $B_{\mathbf{n}} = \mathrm{D}^2 S_{\mathbf{n}}$, $\zeta_{\mathbf{n}} = S_{\mathbf{n}} / \sqrt{B_{\mathbf{n}}}$ and let $\mu_{\zeta_{\mathbf{n}}}$ denote the distribution of the r.v. $\zeta_{\mathbf{n}}$.

The following two theorems are extensions of the results on independent $X_{\mathbf{n}}$'s in Fazekas and Rychlik (2001) [6].

Tómács (2002) [13] Theorem 2.1.

Theorem 4.2. *Let $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ be an m-dependent random field, $\mathrm{E}X_{\mathbf{n}} = 0$, $\mathbf{n} \in \mathbb{N}^d$. Suppose that*

$$\text{there exist } M, \delta \ge 0, \text{ such that } \mathrm{E}|X_{\mathbf{n}}|^{2+\delta} \le M < \infty \text{ for all } \mathbf{n} \in \mathbb{N}^d \tag{4.1}$$

and

$$\text{there exist } \sigma > 0 \text{ and } \mathbf{n}_\sigma \in \mathbb{N}^d \text{ such that } \frac{B_{\mathbf{n}}}{|\mathbf{n}|} \ge \sigma \text{ for all } \mathbf{n} \ge \mathbf{n}_\sigma. \tag{4.2}$$

Let $0 \le d_k^{(i)} \le c \log \frac{k+1}{k}$, assume that $\sum_{k=1}^{\infty} d_k^{(i)} = \infty$ for all $i \in \{1, \dots, d\}$. Let $d_{\mathbf{k}} = \prod_{i=1}^d d_{k_i}^{(i)}$ and $D_{\mathbf{n}} = \sum_{\mathbf{k} \le \mathbf{n}} d_{\mathbf{k}}$. Then for any probability distribution μ the following two statements are equivalent:

$$\frac{1}{D_{\mathbf{n}}} \sum_{\mathbf{k} \le \mathbf{n}} d_{\mathbf{k}} \delta_{\zeta_{\mathbf{k}}(\omega)} \Rightarrow \mu \ (\mathbf{n} \to \infty) \text{ for almost every } \omega \in \Omega,$$

$$\frac{1}{D_{\mathbf{n}}} \sum_{\mathbf{k} \le \mathbf{n}} d_{\mathbf{k}} \mu_{\zeta_{\mathbf{k}}} \Rightarrow \mu \ (\mathbf{n} \to \infty).$$

Tómács (2002) [13] Theorem 2.2.

Theorem 4.3. *Let $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ be an m-dependent random field, $\mathrm{E}X_{\mathbf{n}} = 0$, $\mathbf{n} \in \mathbb{N}^d$. Assume that (4.1) and (4.2) hold for some $\delta > 0$. Then*

$$\frac{1}{|\log \mathbf{n}|} \sum_{\mathbf{k} \le \mathbf{n}} \frac{1}{|\mathbf{k}|} \delta_{\zeta_{\mathbf{k}}(\omega)} \Rightarrow \mathcal{N}(0,1) \ (\mathbf{n} \to \infty) \quad \text{for almost every } \omega \in \Omega.$$

5. On the Rosenthal inequality for mixing fields

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two σ-algebras in $\mathcal{F}$. The *α-mixing coefficient* is defined as follows.

$$\alpha(\mathcal{F}_1, \mathcal{F}_2) := \sup\{|\mathrm{P}(A)\mathrm{P}(B) - \mathrm{P}(AB)| : A \in \mathcal{F}_1, B \in \mathcal{F}_2\}.$$

Let $d \in \mathbb{N}$ be a fixed positive integer, $I \subseteq \mathbb{Z}^d$, $\|\mathbf{n}\| = \max\{|n_1|, \ldots, |n_d|\}$, where $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$, and $\varrho(\mathbf{n}, \mathbf{m}) = \|\mathbf{n} - \mathbf{m}\|$. Let $Y = \{Y_\mathbf{t} : \mathbf{t} \in I\}$ be a multiindex sequence of r.v.'s. The *α-mixing coefficient of* Y is

$$\alpha_Y(r, u, v) := \sup\{\alpha(\mathcal{F}_{I_1}, \mathcal{F}_{I_2}) : \varrho(I_1, I_2) \geq r,\ \mathrm{card}(I_1) \leq u,\ \mathrm{card}(I_2) \leq v\},$$

where I_1 and I_2 are finite subsets in I, $\mathcal{F}_{I_i} = \sigma\{Y_\mathbf{t} : \mathbf{t} \in I_i\}$, $i = 1, 2$.

Let T be a finite set in I. Introduce the following notation.

$$L(\mu, \varepsilon, T) = \sum_{\mathbf{t}\in T} \left(\mathrm{E}|Y_\mathbf{t}|^{\mu+\varepsilon}\right)^{\mu/(\mu+\varepsilon)} = \sum_{\mathbf{t}\in T} \|Y_\mathbf{t}\|^{\mu}_{\mu+\varepsilon}.$$

$$D(h, \varepsilon, T) = \begin{cases} L(h, 0, T), & \text{if } 0 < h \leq 1, \\ L(h, \varepsilon, T), & \text{if } 1 < h \leq 2, \\ \max\{L(h, \varepsilon, T), L^{h/2}(2, \varepsilon, T)\}, & \text{if } 2 < h. \end{cases}$$

Let $s_r = \mathrm{card}(\{\mathbf{t} \in \mathbb{Z}^d : \|\mathbf{t}\| = r\} \cap I)$, $b_r = \mathrm{card}(\{\mathbf{t} \in \mathbb{Z}^d : \|\mathbf{t}\| \leq r\} \cap I)$ and

$$c^{(\alpha)}_{u,h-u} := 8u!(h-u-1)!(h-1)! \sum_{r=1}^{\infty} \left(\alpha_Y(r, u, h-u)\right)^{\varepsilon/(h+\varepsilon)} s_r b_r^{h-2}.$$

The following theorem is a version of Theorem 1 of Doukhan (1994) [1]. In the proof of that theorem we have found a gap. For our version we gave detailed proof.

Fazekas, Kukush and Tómács [5] Theorem.

Theorem 5.1. *Let $l > 1$ and $\varepsilon > 0$. Let $\{Y_\mathbf{t}, \mathbf{t} \in I\}$ be centered random variables with $\mathrm{E}|Y_\mathbf{t}|^{l+\varepsilon} < \infty$, $\mathbf{t} \in I$. Let h be the smallest even integer with $h \geq l$. Assume that $c^{(\alpha)}_{u,h-u} < \infty$ for $u = 1, \ldots, h-1$. Then there is a constant $K_{(\alpha)}$ such that*

$$\mathrm{E}\left|\sum_{\mathbf{t}\in T} Y_\mathbf{t}\right|^l \leq K_{(\alpha)} D(l, \varepsilon, T) \tag{5.1}$$

for any finite subset T of I.

Remark 5.2. $K_{(\alpha)}$ does not depend on T but it depends on the mixing coefficients and l. If $0 < \varepsilon < l/2$, then $K_{(\alpha)} = H_h^{(\alpha)} C_l$, where

$$H_h^{(\alpha)} = 1 + \sum_{u=1}^{h-1} c_{u,h-u}^{(\alpha)} + \sum_{u=2}^{h-2} \binom{h}{u} H_u^{(\alpha)} H_{h-u}^{(\alpha)},$$
$$C_l = 2^{(h-l+\varepsilon)(2h+2l-1)/\varepsilon}.$$

If l is an even integer, then one can put $C_l = 1$. Inequality (5.1) is always satisfied for $0 < l \leq 1$, if we replace $K_{(\alpha)}$ with 1.

6. References

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