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An almost sure limit theorem for α-mixing random fields

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Abstract

An almost sure limit theorem with logarithmic averages for α-mixing random fields is presented.

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MSC: 60F15, 60F17

1. Introduction

Let $\mathbb{N}$ be the set of the positive integers, $\mathbb{R}$ the set of real numbers and $\mathcal{B}$ the σ-algebra of Borel sets of $\mathbb{R}$. Let δ_x be the unit mass at point x, that is $\delta_x\colon \mathcal{B} \to \mathbb{R}$, $\delta_x(B) = 1$ if $x \in B$ and $\delta_x(B) = 0$ if $x \notin B$. Denote $\xrightarrow{\mathrm{w}} \mu$ the weak convergence to the probability measure μ. In the following all random variables defined on a fixed probability space $(\Omega, \mathcal{F}, \mathrm{P})$. Almost sure (a.s.) limit theorems state that

$$\frac{1}{D_n}\sum_{k=1}^{n} d_k \delta_{\zeta_k(\omega)} \xrightarrow{\mathrm{w}} \mu \quad \text{as} \quad n \to \infty, \quad \text{for almost every} \quad \omega \in \Omega,$$

where ζ_k $(k \in \mathbb{N})$ are random variables. The simplest form of it is the so-called classical a.s. central limit theorem, in which $\zeta_k = (X_1 + \cdots + X_k)/\sqrt{k}$, where $X_1, X_2, \ldots$ are independent identically distributed (i.i.d.) random variables with expectation 0 and variance 1, moreover $d_k = 1/k$, $D_n = \log n$ and μ is the standard normal distribution $\mathcal{N}(0,1)$. (See Berkes [1] for an overview.)

Let $\mathbb{N}^d$ be the positive integer d-dimensional lattice points, where d is a fixed positive integer. In this paper $\mathbf{k} = (k_1, \ldots, k_d), \mathbf{n} = (n_1, \ldots, n_d), \ldots \in \mathbb{N}^d$. Relations $\leqslant$, $\not<$, min, max, $\to$ etc. are defined coordinatewise, i.e. $\mathbf{n} \to \infty$ means that

$n_i \to \infty$ for all $i \in \{1, \ldots, d\}$. Let $|\mathbf{n}| = \prod_{i=1}^d n_i$ and $|\log \mathbf{n}| = \prod_{i=1}^d \log_+ n_i$, where $\log_+ x = \log x$ if $x \geqslant e$ and $\log_+ x = 1$ if $x < e$. The general form of the multiindex version of the a.s. limit theorems is

$$\frac{1}{D_{\mathbf{n}}} \sum_{\mathbf{k} \leqslant \mathbf{n}} d_{\mathbf{k}} \delta_{\zeta_{\mathbf{k}}(\omega)} \xrightarrow{\text{w}} \mu \quad \text{as} \quad \mathbf{n} \to \infty, \quad \text{for almost every} \quad \omega \in \Omega,$$

where $\{\zeta_{\mathbf{k}}, \mathbf{k} \in \mathbb{N}^d\}$ is a random field (multiindex sequence of random variables). In the multiindex version of the classical a.s. central limit theorem $X_{\mathbf{i}}, \mathbf{i} \in \mathbb{N}^d$ i.i.d. random variables with expectation 0 and variance 1, $\zeta_{\mathbf{k}} = \sum_{\mathbf{i} \leqslant \mathbf{k}} X_{\mathbf{i}} / \sqrt{|\mathbf{k}|}$, $d_{\mathbf{k}} = 1/|\mathbf{k}|$, $D_{\mathbf{n}} = 1/|\log \mathbf{n}|$ and $\mu = \mathcal{N}(0,1)$. It is well-known that generally the multiindex cases are not direct consequences of the corresponding theorems for ordinary sequences.

Fazekas and Rychlik proved in [5] a general a.s. limit theorem for multiindex sequences of metric space valued random elements. Tómács proved in [8] an a.s. central limit theorem for m-dependent random fields. In this paper we shall prove an a.s. limit theorem with logarithmic averages for α-mixing random fields (Theorem 2.5). Its onedimension version for $\mu = \mathcal{N}(0,1)$ is proved by Fazekas and Rychlik (see [4, Proposition 3.2]). In the proof of Theorem 2.5 we shall use a multiindex strong law of large numbers (Theorem 2.1). In the proof of Theorem 2.3 we shall follow ideas of Berkes and Csáki [2].

Throughout the paper we use the following notation. Let $\mathbb{R}_+$ be the set of the positive real numbers. If $a_1, a_2, \ldots \in \mathbb{R}$ then in case $A = \emptyset$ let $\max_{k \in A} a_k = 0$ and $\sum_{k \in A} a_k = 0$. Let $[A]$ be the closure of $A \subset \mathbb{R}$ and $\partial A = [A] \cap [\,\overline{A}\,]$.

If ξ is a random variable, then let μ_ξ denote the distribution of ξ, $\|\xi\|_\infty = \inf\{c \in \mathbb{R} : \mathrm{P}(|\xi| \leqslant c) = 1\}$ and $\sigma(\xi) = \{\xi^{-1}(B) : B \in \mathcal{B}\}$.

In the following let $\{c_k^{(i)} \in \mathbb{R}_+, k \in \mathbb{N}\}$ be increasing sequences with $c_{k+1}^{(i)}/c_k^{(i)} = O(1)$, $\lim_{n\to\infty} c_n^{(i)} = \infty$ for each $i = 1, \ldots, d$, and the sequences $\{d_k^{(i)} \in \mathbb{R}_+, k \in \mathbb{N}\}$ have the next properties: $d_k^{(i)} \leqslant \log(c_{k+1}^{(i)}/c_k^{(i)})$ for all $k \in \mathbb{N}$ and $i = 1, \ldots, d$, moreover $\sum_{k=1}^\infty d_k^{(i)} = \infty$ for each $i = 1, \ldots, d$. Let $d_{\mathbf{k}} = \prod_{i=1}^d d_{k_i}^{(i)}$, $D_{\mathbf{n}} = \sum_{\mathbf{k} \leqslant \mathbf{n}} d_{\mathbf{k}}$ and $D_{n_i}^{(i)} = \sum_{k=1}^{n_i} d_k^{(i)}$.

2. Results

Theorem 2.1. *Let $\{\xi_{\mathbf{i}}, \mathbf{i} \in \mathbb{N}^d\}$ be a uniformly bounded random field, namely there exists $c \in \mathbb{R}_+$ such that $|\xi_{\mathbf{i}}| \leqslant c$ a.s. for all $\mathbf{i} \in \mathbb{N}^d$. Assume that there exist $c_1, c_2, \varepsilon \in \mathbb{R}_+$ and $\alpha_{\mathbf{k},\mathbf{l}} \in \mathbb{R}$ $(\mathbf{k}, \mathbf{l} \in \mathbb{N}^d)$ such that*

$$\sum_{\mathbf{l} \leqslant \mathbf{n}} \sum_{\mathbf{k} \leqslant \mathbf{n}} d_{\mathbf{k}} d_{\mathbf{l}} \alpha_{\mathbf{k},\mathbf{l}} \leqslant c_1 D_{\mathbf{n}}^2 \prod_{i=1}^d \left(\log D_{n_i}^{(i)}\right)^{-1-\varepsilon} \tag{2.1}$$

for all enough large $n_i \in \mathbb{N}$, and

$$|\mathrm{E}\,\xi_{\mathbf{k}}\xi_{\mathbf{l}}| \leqslant c_2 \left(\prod_{i=1}^{d} \left(\log_+ \log_+ \frac{c_{m_i}^{(i)}}{c_{h_i}^{(i)}} \right)^{-1-\varepsilon} + \alpha_{\mathbf{k},\mathbf{l}} \right) \tag{2.2}$$

for each $\mathbf{k}, \mathbf{l} \in \mathbb{N}^d$, where $\mathbf{h} = \min\{\mathbf{k}, \mathbf{l}\}$ and $\mathbf{m} = \max\{\mathbf{k}, \mathbf{l}\}$. Then

$$\frac{1}{D_{\mathbf{n}}} \sum_{\mathbf{k} \leqslant \mathbf{n}} d_{\mathbf{k}} \xi_{\mathbf{k}} \to 0 \quad \textit{as} \quad \mathbf{n} \to \infty \quad \textit{a.s.}$$

Definition 2.2. The *α-mixing coefficient* of the random variables ξ and η is

$$\alpha(\xi, \eta) = \alpha\big(\sigma(\xi), \sigma(\eta)\big) = \sup_{\substack{A \in \sigma(\xi) \\ B \in \sigma(\eta)}} |\,\mathrm{P}(AB) - \mathrm{P}(A)\,\mathrm{P}(B)|.$$

Theorem 2.3. *Let $\{\zeta_{\mathbf{k}}, \mathbf{k} \in \mathbb{N}^d\}$ be a random field. Assume that there exist random variables $\zeta_{\mathbf{h},\mathbf{l}}$ $(\mathbf{h} \leqslant \mathbf{l})$ and $c_1, c_2, c_3, \varepsilon \in \mathbb{R}_+$ such that*

$$|\zeta_{\mathbf{k}} - \zeta_{\mathbf{h},\mathbf{k}}| \geqslant c_1 \quad \textit{a.s.} \quad \forall \mathbf{h}, \mathbf{k} \in \mathbb{N}^d \quad \textit{for which} \quad \mathbf{h} \leqslant \mathbf{k}, \tag{2.3}$$

$$\mathrm{E} \min \left\{ (\zeta_{\mathbf{l}} - \zeta_{\mathbf{h},\mathbf{l}})^2, 1 \right\} \leqslant c_2 \prod_{i=1}^{d} \left(\log_+ \log_+ \frac{c_{l_i}^{(i)}}{c_{h_i}^{(i)}} \right)^{-2-2\varepsilon} \tag{2.4}$$

for all $\mathbf{h}, \mathbf{l} \in \mathbb{N}^d$ for which $\mathbf{h} \leqslant \mathbf{l}$, and

$$\sum_{\mathbf{l} \leqslant \mathbf{n}} \sum_{\mathbf{k} \leqslant \mathbf{n}} d_{\mathbf{k}} d_{\mathbf{l}} \alpha_{\mathbf{k},\mathbf{l}} \leqslant c_3 D_{\mathbf{n}}^2 \prod_{i=1}^{d} \left(\log D_{n_i}^{(i)} \right)^{-1-\varepsilon} \tag{2.5}$$

for all enough large $n_i \in \mathbb{N}$, where $\alpha_{\mathbf{k},\mathbf{l}} = \alpha(\zeta_{\mathbf{k}}, \zeta_{\mathbf{t},\mathbf{l}})$ with $\mathbf{t} = \min\{\mathbf{k}, \mathbf{l}\}$. Then for any probability distribution μ the following two statements are equivalent:

(1) $\displaystyle \frac{1}{D_{\mathbf{n}}} \sum_{\mathbf{k} \leqslant \mathbf{n}} d_{\mathbf{k}} \delta_{\zeta_{\mathbf{k}}(\omega)} \xrightarrow{\mathrm{w}} \mu$ *as $\mathbf{n} \to \infty$, for almost every $\omega \in \Omega$;*

(2) $\displaystyle \frac{1}{D_{\mathbf{n}}} \sum_{\mathbf{k} \leqslant \mathbf{n}} d_{\mathbf{k}} \mu_{\zeta_{\mathbf{k}}} \xrightarrow{\mathrm{w}} \mu$ *as $\mathbf{n} \to \infty$.*

Definition 2.4. The *α-mixing coefficient* of the random field $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ is

$$\alpha(\mathbf{k}) = \sup_{\mathbf{n}} \alpha \left(\bigcup_{\mathbf{i} \leqslant \mathbf{n}} \sigma(X_{\mathbf{i}}), \bigcup_{\mathbf{i} \not\leqslant \mathbf{n}+\mathbf{k}} \sigma(X_{\mathbf{i}}) \right), \quad \mathbf{k} \in \mathbb{N}^d.$$

Theorem 2.5. *Let $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ be an α-mixing random field with mixing coefficient*

$$\alpha(\mathbf{k}) \leqslant \frac{c}{|\log \mathbf{k}|} \tag{2.6}$$

for all $\mathbf{k} \in \mathbb{N}^d$, where $c \in \mathbb{R}_+$ is fixed. Let $S_{\mathbf{n}} = \sum_{\mathbf{k} \leqslant \mathbf{n}} X_{\mathbf{k}}$ and $\sigma_{\mathbf{n}}^2 = \mathrm{E}\, S_{\mathbf{n}}^2 > 0$. Assume that $\mathrm{E}\, X_{\mathbf{i}} = 0$ and $\mathrm{E}\, X_{\mathbf{i}}^2 < \infty$ for all $\mathbf{i} \in \mathbb{N}^d$, moreover there exist $c_1, c_2 \in \mathbb{R}_+$ and $\beta > 2/\log 2$ such that

$$|S_{\mathbf{l}}| \geqslant c_1 \sigma_{\mathbf{k}} \quad \textit{a.s.} \quad \forall \mathbf{l}, \mathbf{k} \in \mathbb{N}^d \quad \textit{for which} \quad \mathbf{l} \leqslant \mathbf{k} \tag{2.7}$$

and

$$\mathrm{E} \min\left\{\frac{S_{\mathbf{r}}^2}{\sigma_{\mathbf{l}}^2}, 1\right\} \leqslant c_2 \left(\frac{|\mathbf{h}|}{|\mathbf{l}|}\right)^{\beta} \quad \forall \mathbf{h}, \mathbf{l} \in \mathbb{N}^d \quad \textit{for which} \quad \mathbf{h} \leqslant \mathbf{l}, \tag{2.8}$$

where $\mathbf{r} = 2\mathbf{h}$ if $2\mathbf{h} < \mathbf{l}$ and $\mathbf{r} = \mathbf{l}$ otherwise. If $\mu_{\zeta_{\mathbf{n}}} \xrightarrow{\mathrm{w}} \mu$ as $\mathbf{n} \to \infty$, where $\zeta_{\mathbf{n}} = S_{\mathbf{n}}/\sigma_{\mathbf{n}}$ and μ is a probability distribution, then

$$\frac{1}{|\log \mathbf{n}|} \sum_{\mathbf{k} \leqslant \mathbf{n}} \frac{1}{|\mathbf{k}|} \delta_{\zeta_{\mathbf{k}}(\omega)} \xrightarrow{\mathrm{w}} \mu \quad \textit{as} \quad \mathbf{n} \to \infty, \quad \textit{for almost every} \quad \omega \in \Omega.$$

3. Lemmas

You can find the proof of the next lemma in [6].

Lemma 3.1 (Covariance inequality). *If ξ and η are bounded random variables, then*

$$|\operatorname{cov}(\xi, \eta)| \leqslant 4\alpha(\xi, \eta) \|\xi\|_\infty \|\eta\|_\infty.$$

The proof of the next lemma follows from that of Theorem 11.3.3 and Corollary 11.3.4 in [3].

Lemma 3.2. *Let BL denote the set of all bounded, real-valued Lipshitz function on $\mathbb{R}$. If μ and μ_n are distributions ($n \in \mathbb{N}$), then there exists a countable set $M \subset BL$ (depending on μ) such that the following are equivalent:*

(1) $\mu_n \xrightarrow{\mathrm{w}} \mu$ *as* $n \to \infty$;

(2) $\int g \, \mathrm{d}\mu_n \to \int g \, \mathrm{d}\mu$ *as* $n \to \infty$ *for all* $g \in M$.

Lemma 3.3 (Theorem 1 of [7], p. 309). *If μ and μ_n are distributions ($n \in \mathbb{N}$), then the following are equivalent:*

(1) $\mu_n \xrightarrow{\mathrm{w}} \mu$ *as* $n \to \infty$;

(2) $\mu_n(A) \to \mu(A)$ *as* $n \to \infty$ *for all* $A \in \mathcal{B}$ *for which* $\mu(\partial A) = 0$.

Lemma 3.4. *If μ and $\mu_{\mathbf{n}}$ are distributions ($\mathbf{n} \in \mathbb{N}^d$) and $\mu_{\mathbf{n}} \xrightarrow{\mathrm{w}} \mu$ as $\mathbf{n} \to \infty$, then*

$$\frac{1}{D_{\mathbf{n}}} \sum_{\mathbf{k} \leqslant \mathbf{n}} d_{\mathbf{k}} \mu_{\mathbf{k}} \xrightarrow{\mathrm{w}} \mu \quad \textit{as} \quad \mathbf{n} \to \infty.$$

Proof. By $\sum_{k_i=1}^{\infty} d_{k_i}^{(i)} = \infty$ we have

$$\frac{1}{D_{\mathbf{n}}} \sum_{\mathbf{m} \leqslant \mathbf{k} \leqslant \mathbf{n}} d_{\mathbf{k}} = \prod_{i=1}^{d} \frac{\sum_{m_i \leqslant k_i \leqslant n_i} d_{k_i}^{(i)}}{\sum_{k_i \leqslant n_i} d_{k_i}^{(i)}} \to 1 \quad \text{as} \quad \mathbf{n} \to \infty \quad \forall \mathbf{m} \in \mathbb{N}^d,$$

which implies, that

$$\frac{1}{D_{\mathbf{n}}} \sum_{\substack{\mathbf{k} \leqslant \mathbf{n} \\ \mathbf{k} \ngeqslant \mathbf{m}}} d_{\mathbf{k}} = 1 - \frac{1}{D_{\mathbf{n}}} \sum_{\mathbf{m} \leqslant \mathbf{k} \leqslant \mathbf{n}} d_{\mathbf{k}} \to 0 \quad \text{as} \quad \mathbf{n} \to \infty \quad \forall \mathbf{m} \in \mathbb{N}^d. \tag{3.1}$$

Let $f \colon \mathbb{R} \to \mathbb{R}$ be a bounded and continuous function and $K = \sup_{x \in \mathbb{R}} |f(x)|$. Then

$$\left| \int f \, \mathrm{d}\mu_{\mathbf{n}} - \int f \, \mathrm{d}\mu \right| \leqslant \int K \, \mathrm{d}\mu_{\mathbf{n}} + \int K \, \mathrm{d}\mu = 2K, \tag{3.2}$$

moreover by $\mu_{\mathbf{n}} \xrightarrow{\mathrm{w}} \mu$ and (3.1), for any $\varepsilon > 0$ there exists $\mathbf{n}(\varepsilon) \in \mathbb{N}^d$ such that

$$\left| \int f \, \mathrm{d}\mu_{\mathbf{n}} - \int f \, \mathrm{d}\mu \right| < \frac{\varepsilon}{2} \tag{3.3}$$

and

$$\frac{1}{D_{\mathbf{n}}} \sum_{\substack{\mathbf{k} \leqslant \mathbf{n} \\ \mathbf{k} \ngeqslant \mathbf{n}(\varepsilon)}} d_{\mathbf{k}} < \frac{\varepsilon}{4K} \tag{3.4}$$

for all $\mathbf{n} \geqslant \mathbf{n}(\varepsilon)$. With notation $\gamma_{\mathbf{n}} = \frac{1}{D_{\mathbf{n}}} \sum_{\mathbf{k} \leqslant \mathbf{n}} d_{\mathbf{k}} \mu_{\mathbf{k}}$ the inequalities (3.2), (3.3) and (3.4) imply, that

$$\begin{aligned}
&\left| \int f \, \mathrm{d}\gamma_{\mathbf{n}} - \int f \, \mathrm{d}\mu \right| \leqslant \frac{1}{D_{\mathbf{n}}} \sum_{\mathbf{k} \leqslant \mathbf{n}} d_{\mathbf{k}} \left| \int f \, \mathrm{d}\mu_{\mathbf{k}} - \int f \, \mathrm{d}\mu \right| \\
&= \frac{1}{D_{\mathbf{n}}} \sum_{\substack{\mathbf{k} \leqslant \mathbf{n} \\ \mathbf{k} \ngeqslant \mathbf{n}(\varepsilon)}} d_{\mathbf{k}} \left| \int f \, \mathrm{d}\mu_{\mathbf{k}} - \int f \, \mathrm{d}\mu \right| + \frac{1}{D_{\mathbf{n}}} \sum_{\mathbf{n}(\varepsilon) \leqslant \mathbf{k} \leqslant \mathbf{n}} d_{\mathbf{k}} \left| \int f \, \mathrm{d}\mu_{\mathbf{k}} - \int f \, \mathrm{d}\mu \right| \\
&< \frac{1}{D_{\mathbf{n}}} \sum_{\substack{\mathbf{k} \leqslant \mathbf{n} \\ \mathbf{k} \ngeqslant \mathbf{n}(\varepsilon)}} d_{\mathbf{k}} \cdot 2K + \frac{1}{D_{\mathbf{n}}} \sum_{\mathbf{n}(\varepsilon) \leqslant \mathbf{k} \leqslant \mathbf{n}} d_{\mathbf{k}} \cdot \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

for all $\mathbf{n} \geqslant \mathbf{n}(\varepsilon)$. This fact implies the statement. □

4. Proof of the theorems

Proof of Theorem 2.1. By (2.2) and (2.1) we have

$$\mathrm{E} \left(\sum_{\mathbf{k} \leqslant \mathbf{n}} d_{\mathbf{k}} \xi_{\mathbf{k}} \right)^2 \leqslant \sum_{\mathbf{k} \leqslant \mathbf{n}} \sum_{\mathbf{l} \leqslant \mathbf{n}} d_{\mathbf{k}} d_{\mathbf{l}} \left| \mathrm{E}\, \xi_{\mathbf{k}} \xi_{\mathbf{l}} \right|$$

$$\leqslant c_2 \sum_{\mathbf{k}\leqslant\mathbf{n}} \sum_{\mathbf{l}\leqslant\mathbf{n}} \prod_{i=1}^{d} d_{k_i}^{(i)} d_{l_i}^{(i)} \left(\log_+ \log_+ \frac{c_{m_i}^{(i)}}{c_{h_i}^{(i)}} \right)^{-1-\varepsilon} + c_2 \sum_{\mathbf{k}\leqslant\mathbf{n}} \sum_{\mathbf{l}\leqslant\mathbf{n}} d_{\mathbf{k}} d_{\mathbf{l}} \alpha_{\mathbf{k},\mathbf{l}}$$

$$\leqslant 2c_2 \prod_{i=1}^{d} \sum_{k_i\leqslant l_i\leqslant n_i} d_{k_i}^{(i)} d_{l_i}^{(i)} \left(\log_+ \log_+ \frac{c_{l_i}^{(i)}}{c_{k_i}^{(i)}} \right)^{-1-\varepsilon} + c_2 c_1 D_{\mathbf{n}}^2 \prod_{i=1}^{d} \left(\log D_{n_i}^{(i)} \right)^{-1-\varepsilon} \tag{4.1}$$

for all enough large n_i. Now assume that $(k_i, l_i) \in A_{n_i}^{(i)}$, where

$$A_{n_i}^{(i)} = \left\{ (k_i, l_i) : k_i \leqslant l_i \leqslant n_i \text{ and } c_{l_i}^{(i)}/c_{k_i}^{(i)} \geqslant \exp\left(\sqrt{D_{n_i}^{(i)}}\right) \right\}.$$

Then $\log_+ \log_+ \left(c_{l_i}^{(i)}/c_{k_i}^{(i)} \right) \geqslant \frac{1}{2} \log D_{n_i}^{(i)}$, which implies, that

$$\sum_{(k_i,l_i)\in A_{n_i}^{(i)}} d_{k_i}^{(i)} d_{l_i}^{(i)} \left(\log_+ \log_+ \frac{c_{l_i}^{(i)}}{c_{k_i}^{(i)}} \right)^{-1-\varepsilon}$$

$$\leqslant 2^{1+\varepsilon} \left(\log D_{n_i}^{(i)} \right)^{-1-\varepsilon} \sum_{(k_i,l_i)\in A_{n_i}^{(i)}} d_{k_i}^{(i)} d_{l_i}^{(i)} \leqslant 2^{1+\varepsilon} \left(D_{n_i}^{(i)} \right)^2 \left(\log D_{n_i}^{(i)} \right)^{-1-\varepsilon}. \tag{4.2}$$

If $(k_i, l_i) \in B_{n_i}^{(i)}$, where

$$B_{n_i}^{(i)} = \left\{ (k_i, l_i) : k_i \leqslant l_i \leqslant n_i \text{ and } c_{l_i}^{(i)}/c_{k_i}^{(i)} < \exp\left(\sqrt{D_{n_i}^{(i)}}\right) \right\},$$

then with notation $M_i = \sup_k (c_{k+1}^{(i)}/c_k^{(i)})$, we get

$$\log \frac{c_{l_i+1}^{(i)}}{c_{k_i}^{(i)}} = \log \frac{c_{l_i+1}^{(i)}}{c_{l_i}^{(i)}} + \log \frac{c_{l_i}^{(i)}}{c_{k_i}^{(i)}} < \log M_i + \sqrt{D_{n_i}^{(i)}}.$$

Thus we have the following inequality, where $B_{n_i,k_i}^{(i)} = \left\{ l_i : (k_i, l_i) \in B_{n_i}^{(i)} \right\}$.

$$\sum_{(k_i,l_i)\in B_{n_i}^{(i)}} d_{k_i}^{(i)} d_{l_i}^{(i)} \left(\log_+ \log_+ \frac{c_{l_i}^{(i)}}{c_{k_i}^{(i)}} \right)^{-1-\varepsilon} \leqslant \sum_{(k_i,l_i)\in B_{n_i}^{(i)}} d_{k_i}^{(i)} d_{l_i}^{(i)}$$

$$\leqslant \sum_{(k_i,l_i)\in B_{n_i}^{(i)}} d_{k_i}^{(i)} \log \frac{c_{l_i+1}^{(i)}}{c_{l_i}^{(i)}} = \sum_{k_i=1}^{n_i} \sum_{l_i\in B_{n_i,k_i}^{(i)}} d_{k_i}^{(i)} \log \frac{c_{l_i+1}^{(i)}}{c_{l_i}^{(i)}} \leqslant \sum_{k_i=1}^{n_i} d_{k_i}^{(i)} \sum_{l_i=k_i}^{\max B_{n_i,k_i}^{(i)}} \log \frac{c_{l_i+1}^{(i)}}{c_{l_i}^{(i)}}$$

$$= \sum_{k_i=1}^{n_i} d_{k_i}^{(i)} \log \prod_{l_i=k_i}^{\max B_{n_i,k_i}^{(i)}} \frac{c_{l_i+1}^{(i)}}{c_{l_i}^{(i)}} = \sum_{k_i=1}^{n_i} d_{k_i}^{(i)} \log \frac{c^{(i)}_{\max B_{n_i,k_i}^{(i)}}}{c_{k_i}^{(i)}}$$

$$< \sum_{k_i=1}^{n_i} d_{k_i}^{(i)} \left(\log M_i + \sqrt{D_{n_i}^{(i)}} \right) \leqslant \sum_{k_i=1}^{n_i} d_{k_i}^{(i)} 2\sqrt{D_{n_i}^{(i)}} = 2 \left(D_{n_i}^{(i)} \right)^{3/2}$$

for all enough large n_i. It follows from this inequality and (4.2) that

$$\begin{aligned}
&\sum_{k_i\leqslant l_i\leqslant n_i} d_{k_i}^{(i)} d_{l_i}^{(i)} \left(\log_+ \log_+ \frac{c_{l_i}^{(i)}}{c_{k_i}^{(i)}}\right)^{-1-\varepsilon} \\
&\leqslant 2^{1+\varepsilon} \left(D_{n_i}^{(i)}\right)^2 \left(\log D_{n_i}^{(i)}\right)^{-1-\varepsilon} + 2\left(D_{n_i}^{(i)}\right)^{3/2} \\
&\leqslant 2^{1+\varepsilon} \left(D_{n_i}^{(i)}\right)^2 \left(\left(\log D_{n_i}^{(i)}\right)^{-1-\varepsilon} + \left(D_{n_i}^{(i)}\right)^{-1/2}\right) \\
&\leqslant 2^{2+\varepsilon} \left(D_{n_i}^{(i)}\right)^2 \left(\log D_{n_i}^{(i)}\right)^{-1-\varepsilon} \qquad (4.3)
\end{aligned}$$

for all enough large n_i. In the last step we use the inequality $(D_{n_i}^{(i)})^{-1/2} \leqslant (\log D_{n_i}^{(i)})^{-1-\varepsilon}$, which follows from $(D_{n_i}^{(i)})^{1/2}/(\log D_{n_i}^{(i)})^{1+\varepsilon} \to \infty$ as $n_i \to \infty$. By (4.1) and (4.3) we get

$$\mathrm{E}\left(\sum_{\mathbf{k}\leqslant\mathbf{n}} d_{\mathbf{k}}\xi_{\mathbf{k}}\right)^2 \leqslant \text{const.} \prod_{i=1}^{d} \left(D_{n_i}^{(i)}\right)^2 \left(\log D_{n_i}^{(i)}\right)^{-1-\varepsilon} \qquad (4.4)$$

for all enough large n_i. Let

$$n_i(t) = \min\left\{n_i : D_{n_i}^{(i)} \leqslant \exp\left(t^{\frac{1+\varepsilon/2}{1+\varepsilon}}\right)\right\}$$

and $\mathbf{n}(\mathbf{t}) = \big(n_1(t_1), \dots, n_d(t_d)\big)$. Since $n_i(t_i) \to \infty$ as $t_i \to \infty$, thus by (4.4) there exists $\mathbf{T} \in \mathbb{N}^d$, such that

$$\begin{aligned}
&\mathrm{E}\sum_{\mathbf{t}\geqslant\mathbf{T}} \left(\frac{1}{D_{\mathbf{n}(\mathbf{t})}} \sum_{\mathbf{k}\leqslant\mathbf{n}(\mathbf{t})} d_{\mathbf{k}}\xi_{\mathbf{k}}\right)^2 \leqslant \sum_{\mathbf{t}\geqslant\mathbf{T}} \frac{1}{D^2_{\mathbf{n}(\mathbf{t})}} \text{const.} \prod_{i=1}^{d} \left(D_{n_i(t_i)}^{(i)}\right)^2 \left(\log D_{n_i(t_i)}^{(i)}\right)^{-1-\varepsilon} \\
&\leqslant \sum_{\mathbf{t}\geqslant\mathbf{T}} \frac{1}{D^2_{\mathbf{n}(\mathbf{t})}} \text{const.} \prod_{i=1}^{d} \left(D_{n_i(t_i)}^{(i)}\right)^2 t_i^{-1-\varepsilon/2} = \text{const.} \prod_{i=1}^{d} \sum_{t_i=T_i}^{\infty} t_i^{-1-\varepsilon/2} < \infty,
\end{aligned}$$

which implies

$$\frac{1}{D_{\mathbf{n}(\mathbf{t})}} \sum_{\mathbf{k}\leqslant\mathbf{n}(\mathbf{t})} d_{\mathbf{k}}\xi_{\mathbf{k}} \to 0 \quad \text{as} \quad \mathbf{t} \to \infty \quad \text{a.s.} \qquad (4.5)$$

For all $\mathbf{n} \in \mathbb{N}^d$ there exists $\mathbf{t} \in \mathbb{N}^d$ such that $\mathbf{n}(\mathbf{t}) \leqslant \mathbf{n} \leqslant \mathbf{n}(\mathbf{t}+\mathbf{1})$, where $\mathbf{1} = (1, \dots, 1) \in \mathbb{N}^d$. Thus the uniformly bounding implies

$$\left|\frac{1}{D_{\mathbf{n}}} \sum_{\mathbf{k}\leqslant\mathbf{n}} d_{\mathbf{k}}\xi_{\mathbf{k}}\right| \leqslant \left|\frac{1}{D_{\mathbf{n}(\mathbf{t})}} \sum_{\mathbf{k}\leqslant\mathbf{n}(\mathbf{t})} d_{\mathbf{k}}\xi_{\mathbf{k}}\right| + \frac{1}{D_{\mathbf{n}}} \sum_{\substack{\mathbf{k}\leqslant\mathbf{n} \\ \mathbf{k}\nleqslant\mathbf{n}(\mathbf{t})}} d_{\mathbf{k}}|\xi_{\mathbf{k}}|$$

$$\leqslant \left|\frac{1}{D_{\mathbf{n(t)}}} \sum_{\mathbf{k}\leqslant\mathbf{n(t)}} d_{\mathbf{k}}\xi_{\mathbf{k}}\right| + \frac{1}{D_{\mathbf{n}}} \sum_{\substack{\mathbf{k}\leqslant\mathbf{n}\\ \mathbf{k}\nleqslant\mathbf{n(t)}}} d_{\mathbf{k}}\cdot c$$

$$\leqslant \left|\frac{1}{D_{\mathbf{n(t)}}} \sum_{\mathbf{k}\leqslant\mathbf{n(t)}} d_{\mathbf{k}}\xi_{\mathbf{k}}\right| + c\left(1-\frac{D_{\mathbf{n(t)}}}{D_{\mathbf{n(t+1)}}}\right) \quad \text{a.s.} \tag{4.6}$$

The reader can easy verify that $D_{\mathbf{n(t)}}/D_{\mathbf{n(t+1)}} \to 1$ as $\mathbf{t} \to \infty$, so by (4.5) and (4.6) imply the statement of Theorem 2.1. □

Proof of Theorem 2.3. Let $g \in M$, where M is defined in Lemma 3.2. Then there exists $K \geqslant 1$ such that

$$|g(x)| \leqslant K \quad \text{and} \quad |g(x)-g(y)| \leqslant K|x-y| \quad \forall x,y \in \mathbb{R}. \tag{4.7}$$

We shall prove, that with notation $\xi_{\mathbf{k}} = g(\zeta_{\mathbf{k}}) - \mathrm{E}\, g(\zeta_{\mathbf{k}})$ the conditions of Theorem 2.1 hold true. By (2.5) we get (2.1), moreover by (4.7) we have

$$|\xi_{\mathbf{k}}| \leqslant |g(\zeta_{\mathbf{k}})| + \mathrm{E}\,|g(\zeta_{\mathbf{k}})| \leqslant 2K,$$

thus $\{\xi_{\mathbf{k}}, \mathbf{k} \in \mathbb{N}^d\}$ is a uniformly bounded random field. Now we turn to (2.2). Let $\mathbf{t} = \min\{\mathbf{k}, \mathbf{l}\}$. Lemma 3.1 and (4.7) imply

$$\left|\mathrm{E}\,\xi_{\mathbf{k}}\big(g(\zeta_{\mathbf{t},\mathbf{l}}) - \mathrm{E}\,g(\zeta_{\mathbf{l}})\big)\right| = \left|\operatorname{cov}\big(g(\zeta_{\mathbf{l}}), g(\zeta_{\mathbf{t},\mathbf{l}})\big)\right| \leqslant 4K^2\alpha_{\mathbf{k},\mathbf{l}}. \tag{4.8}$$

On the other hand with notation $\eta_{\mathbf{k},\mathbf{l}} = g(\zeta_{\mathbf{l}}) - g(\zeta_{\mathbf{t},\mathbf{l}})$

$$|\mathrm{E}\,\xi_{\mathbf{k}}\eta_{\mathbf{k},\mathbf{l}}| = \left|\operatorname{cov}\big(g(\zeta_{\mathbf{k}}), \eta_{\mathbf{k},\mathbf{l}}\big)\right| \leqslant \left(\mathrm{E}\,g^2(\zeta_{\mathbf{k}})\,\mathrm{E}\,\eta_{\mathbf{k},\mathbf{l}}^2\right)^{1/2}. \tag{4.9}$$

It is easy to see that $\big(g(x)-g(y)\big)^2 \leqslant 4K^2 \min\big\{(x-y)^2, 1\big\}$, thus

$$\mathrm{E}\,\eta_{\mathbf{k},\mathbf{l}}^2 \leqslant 4K^2 \min\big\{(\zeta_{\mathbf{l}} - \zeta_{\mathbf{t},\mathbf{l}})^2, 1\big\}. \tag{4.10}$$

By (4.7) and (2.3) we have $g^2(\zeta_{\mathbf{k}}) \leqslant K^2(1+1/c_1)^2$ and

$$g^2(\zeta_{\mathbf{k}}) < K^2(c_1+1)^2 = K^2\left(1+\frac{1}{c_1}\right)^2 \cdot c_1^2 \leqslant K^2\left(1+\frac{1}{c_1}\right)^2 (\zeta_{\mathbf{k}} - \zeta_{\mathbf{t},\mathbf{k}})^2,$$

which imply $g^2(\zeta_{\mathbf{k}}) \leqslant \text{const.} \min\big\{(\zeta_{\mathbf{k}} - \zeta_{\mathbf{t},\mathbf{k}})^2, 1\big\}$ a.s. Using this inequality, (4.10), (4.9) and (2.4) we get the following.

$$\begin{aligned}
|\mathrm{E}\,\xi_{\mathbf{k}}\eta_{\mathbf{k},\mathbf{l}}| &\leqslant \text{const.}\left(\mathrm{E}\min\big\{(\zeta_{\mathbf{k}} - \zeta_{\mathbf{t},\mathbf{k}})^2, 1\big\}\,\mathrm{E}\min\big\{(\zeta_{\mathbf{l}} - \zeta_{\mathbf{t},\mathbf{l}})^2, 1\big\}\right)^{1/2} \\
&\leqslant \text{const.}\left(\prod_{i=1}^{d} \log_+\log_+ \frac{c_{k_i}^{(i)}}{c_{t_i}^{(i)}} \cdot \log_+\log_+ \frac{c_{l_i}^{(i)}}{c_{t_i}^{(i)}}\right)^{-1-\varepsilon} \\
&= \text{const.}\left(\prod_{i=1}^{d} \log_+\log_+ \frac{c_{m_i}^{(i)}}{c_{t_i}^{(i)}}\right)^{-1-\varepsilon},
\end{aligned} \tag{4.11}$$

where $\mathbf{m} = \max\{\mathbf{k}, \mathbf{l}\}$. Since $|\operatorname{E}\xi_{\mathbf{k}}\xi_{\mathbf{l}}| \leqslant |\operatorname{E}\xi_{\mathbf{k}}\eta_{\mathbf{k},\mathbf{l}}| + \big|\operatorname{E}\xi_{\mathbf{k}}\big(g(\zeta_{\mathbf{t},\mathbf{l}}) - \operatorname{E}g(\zeta_{\mathbf{l}})\big)\big|$, using (4.11) and (4.8) we have (2.2). Now applying Theorem 2.1 we get

$$\frac{1}{D_{\mathbf{n}}}\sum_{\mathbf{k}\leqslant\mathbf{n}} d_{\mathbf{k}}\xi_{\mathbf{k}} \to 0 \quad \text{as} \quad \mathbf{n}\to\infty \quad \text{a.s.} \tag{4.12}$$

Let $\mu_{\mathbf{n}} = \frac{1}{D_{\mathbf{n}}}\sum_{\mathbf{k}\leqslant\mathbf{n}} d_{\mathbf{k}}\mu_{\zeta_{\mathbf{k}}}$ and $\mu_{\mathbf{n},\omega} = \frac{1}{D_{\mathbf{n}}}\sum_{\mathbf{k}\leqslant\mathbf{n}} d_{\mathbf{k}}\delta_{\zeta_{\mathbf{k}}(\omega)}$ $(\omega\in\Omega)$.

First assume that (2) is true, that is $\mu_{\mathbf{n}} \xrightarrow{\text{w}} \mu$ as $\mathbf{n}\to\infty$. Then Lemma 3.2 implies

$$\int g\,\mathrm{d}\mu_{\mathbf{n}} \to \int g\,\mathrm{d}\mu \quad \text{as} \quad \mathbf{n}\to\infty, \tag{4.13}$$

and (4.12) implies

$$\int g\,\mathrm{d}\mu_{\mathbf{n},\omega} - \int g\,\mathrm{d}\mu_{\mathbf{n}} = \frac{1}{D_{\mathbf{n}}}\sum_{\mathbf{k}\leqslant\mathbf{n}} d_{\mathbf{k}}\xi_{\mathbf{k}}(\omega) \to 0 \tag{4.14}$$

as $\mathbf{n}\to\infty$, for almost every $\omega\in\Omega$. By (4.13) and (4.14) we get $\int g\,\mathrm{d}\mu_{\mathbf{n},\omega} \to \int g\,\mathrm{d}\mu$ as $\mathbf{n}\to\infty$, for almost every $\omega\in\Omega$, thus by Lemma 3.2 we get (1).

Finally assume that (1) is true, that is $\mu_{\mathbf{n},\omega} \xrightarrow{\text{w}} \mu$ as $\mathbf{n}\to\infty$, for almost every $\omega\in\Omega$. Let $A\in\mathcal{B}$ and $\mu(\partial A) = 0$. Then by Lemma 3.3 $\mu_{\mathbf{n},\omega}(A) \to \mu(A)$ as $\mathbf{n}\to\infty$, for almost every $\omega\in\Omega$. It follows that $\mu_{\mathbf{n}}(A) = \int \mu_{\mathbf{n},\omega}(A)\,\mathrm{d}\operatorname{P}(\omega) \to \mu(A)$ as $\mathbf{n}\to\infty$. Thus using Lemma 3.3 we get (2). This completes the proof of Theorem 2.3. □

Proof of Theorem 2.5. Let $d_k^{(i)} = 1/k$, $c_k^{(i)} = k^{1/\log 2}$, $\varepsilon = (\beta\log 2 - 2)/2$, $\zeta_{\mathbf{k},\mathbf{l}} = \zeta_{\mathbf{l}} - S_{2\mathbf{k}}/\sigma_{\mathbf{l}}$ if $2\mathbf{k} < \mathbf{l}$ and $\zeta_{\mathbf{k},\mathbf{l}} = 0$ if $\mathbf{k}\leqslant\mathbf{l}$ and $2\mathbf{k} \not< \mathbf{l}$. We shall prove that conditions of Theorem 2.3 hold. It is easy to see that $\alpha_{\mathbf{k},\mathbf{l}} \leqslant \alpha(\mathbf{k})$ for all $\mathbf{k},\mathbf{l}\in\mathbb{N}^d$, where $\alpha_{\mathbf{k},\mathbf{l}}$ is defined in Theorem 2.3. Therefore by (2.6) we have

$$\begin{aligned}\sum_{\mathbf{l}\leqslant\mathbf{n}}\sum_{\mathbf{k}\leqslant\mathbf{n}} d_{\mathbf{k}}d_{\mathbf{l}}\alpha_{\mathbf{k},\mathbf{l}} &\leqslant \sum_{\mathbf{l}\leqslant\mathbf{n}}\sum_{\mathbf{k}\leqslant\mathbf{n}} \frac{c}{|\mathbf{k}|\cdot|\mathbf{l}|\cdot|\log\mathbf{k}|}\\ &= c\prod_{i=1}^{d}\left(\sum_{k=1}^{n_i}\frac{1}{k\log_+ k}\right)\left(\sum_{l=1}^{n_i}\frac{1}{l}\right).\end{aligned} \tag{4.15}$$

It is well-known that $\sum_{k=1}^{n}\frac{1}{k} \sim \log n$ and $\sum_{k=1}^{n}\frac{1}{k\log_+ k} \sim \log\log n$, where $a_n \sim b_n$ iff $\lim_{n\to\infty} a_n/b_n = 1$. So by (4.15) we have

$$\begin{aligned}\sum_{\mathbf{l}\leqslant\mathbf{n}}\sum_{\mathbf{k}\leqslant\mathbf{n}} d_{\mathbf{k}}d_{\mathbf{l}}\alpha_{\mathbf{k},\mathbf{l}} &\leqslant \text{const.}\prod_{i=1}^{d}\log\log n_i\cdot\log n_i \leqslant \text{const.}\prod_{i=1}^{d}(\log n_i)^2(\log\log n_i)^{-1-\varepsilon}\\ &\leqslant \text{const.}\prod_{i=1}^{d}(\log n_i)^2(\log D_{n_i}^{(i)})^{-1-\varepsilon} \leqslant \text{const.}D_{\mathbf{n}}^2\prod_{i=1}^{d}(\log D_{n_i}^{(i)})^{-1-\varepsilon}\end{aligned}$$

for all enough large n_i, which implies (2.5). Using (2.8)

$$\mathrm{E}\min\left\{(\zeta_{\mathbf{l}}-\zeta_{\mathbf{h},\mathbf{l}})^2,1\right\}=\mathrm{E}\min\left\{S_{\mathbf{r}}^2/\sigma_{\mathbf{l}}^2,1\right\}\leqslant \text{const.}\prod_{i=1}^{d}\left(\log_+\log_+\frac{c_{l_i}^{(i)}}{c_{h_i}^{(i)}}\right)^{-2-2\varepsilon}$$

for all $\mathbf{h},\mathbf{l}\in\mathbb{N}^d$ for which $\mathbf{h}\leqslant\mathbf{l}$, where $\mathbf{r}=2\mathbf{h}$ if $2\mathbf{h}<\mathbf{l}$ and $\mathbf{r}=\mathbf{l}$ if $\mathbf{h}\leqslant\mathbf{l}$ and $2\mathbf{h}\not<\mathbf{l}$, so we get (2.4). The reader can readily verify that (2.3) is hold as well. Now applying Lemma 3.4 and Theorem 2.3, we have

$$\frac{1}{\sum_{\mathbf{k}\leqslant\mathbf{n}}\frac{1}{|\mathbf{k}|}}\sum_{\mathbf{k}\leqslant\mathbf{n}}\frac{1}{|\mathbf{k}|}\delta_{\zeta_{\mathbf{k}}(\omega)}\xrightarrow{\mathrm{w}}\mu\quad\text{as}\quad\mathbf{n}\to\infty,\quad\text{for almost every}\quad\omega\in\Omega.$$

Since $\sum_{\mathbf{k}\leqslant\mathbf{n}}\frac{1}{|\mathbf{k}|}\sim|\log\mathbf{n}|$, we get the statement. $\square$

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