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Convergence rate in the strong law of large numbers for mixingales and superadditive structures

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Abstract

In this paper we study convergence rates in the strong laws of large numbers for mixingales and superadditive structures, by using a general method.

Keywords: convergence rate, strong law of large numbers, L^r mixingale, sequence with superadditive moment function

MSC: 60F15

1. Introduction

Sung, Hu and Volodin [8] introduced a new method for obtaining convergence rate in the strong law of large numbers (SLLN), by using the approach of Fazekas and Klesov [2]. This result generalizes and sharpens the method of Hu and Hu [5]. Tómács [9] gave a general method by using a Hájek–Rényi type inequality (see Hájek and Rényi [3]) for the probabilities, which sharpens the result of Sung, Hu and Volodin [8]. In this paper we apply this method for mixingales and superadditive structures.

The concept of L^2 mixingales was introduced by McLeish [6], and generalized to L^r mixingales by Andrews [1]. The definition of superadditive moment function is due to Móricz [7].

Fazekas and Klesov [2, Theorem 6.1 and 6.2] proved SLLN's for mixingales. In Section 3 we shall give the convergence rates in these SLLN's. Hu and Hu [5, Theorem 2.1] obtained convergence rate in SLLN under the superadditivity property. In Section 4 we shall generalize this result.

We use the following notation. Let $\mathbb{N}$ be the set of the positive integers and $\mathbb{R}$ the set of real numbers. If $a_1, a_2, \ldots \in \mathbb{R}$ then in case $A = \emptyset$ let $\max_{k\in A} a_k = 0$ and

$\sum_{k\in A} a_k = 0$. In this paper let $\{X_k,\ k \in \mathbb{N}\}$ be a sequence of random variables defined on a fixed probability space $(\Omega, \mathcal{F}, \mathrm{P})$, $S_n = \sum_{k=1}^{n} X_k$ for all $n \in \mathbb{N}$ and $S_0 = 0$. Finally in this paper let $\{b_k,\ k \in \mathbb{N}\}$ be a nondecreasing unbounded sequence of positive real numbers.

2. A general method to obtain the rate of convergence in the SLLN

Definition 2.1. Let Θ_r $(r > 0)$ denote the set of functions $\vartheta\colon [0,\infty) \to \mathbb{R}$ which are nondecreasing, continuous at 0, $\vartheta(0) = 0$, $\vartheta(x) > 0$ for all $x > 0$ and

$$\sum_{n=1}^{\infty} n^{-2}\vartheta^{-r}(n^{-1}) < \infty.$$

Remark 2.2. It is easy to see that if $0 < \delta < 1$ and $\vartheta(x) = x^{\delta/r}$ $(x \geqslant 0)$, then $\vartheta \in \Theta_r$.

Theorem 2.3 (Tómács [9], Theorem 3.4). *Let $\{\alpha_k,\ k \in \mathbb{N}\}$ be a sequence of nonnegative real numbers, $r > 0$ and*

$$\beta_n = \max_{k\leqslant n} b_k \vartheta\left(\sum_{i=k}^{\infty} \alpha_i b_i^{-r}\right), \quad \text{where} \quad \vartheta \in \Theta_r. \tag{2.1}$$

If

$$\sum_{k=1}^{\infty} \alpha_k b_k^{-r} < \infty \tag{2.2}$$

and there exists $c > 0$ such that for any $n \in \mathbb{N}$ and any $\varepsilon > 0$

$$\mathrm{P}\Big(\max_{k\leqslant n} |S_k| \geqslant \varepsilon\Big) \leqslant c\varepsilon^{-r} \sum_{k=1}^{n} \alpha_k, \tag{2.3}$$

then

$$\lim_{n\to\infty} \frac{\beta_n}{b_n} = 0 \quad \text{and} \quad \frac{S_n}{b_n} = O\left(\frac{\beta_n}{b_n}\right) \quad \text{almost surely (a.s.).}$$

Lemma 2.4. *Let $\{\alpha_k,\ k \in \mathbb{N}\}$ be a sequence of nonnegative real numbers, $r > 0$, $0 < \delta < 1$, $\vartheta(x) = x^{\delta/r}$ for all $x \geqslant 0$, $b_k = k^{1/r}$ for all $k \in \mathbb{N}$ and let β_n be defined by (2.1). If there exist $c > 0$ and $0 < \gamma < 1$ such that $\sum_{i=k}^{\infty} \alpha_i/i \leqslant c\sum_{i=k}^{\infty} i^{-1-\gamma}$ for all $k \in \mathbb{N}$, then*

$$\frac{\beta_n}{n^{1/r}} = O\left(\frac{1}{n^{\gamma\delta/r}}\right).$$

Proof. Since $\sum_{i=k}^{\infty} i^{-1-\gamma} \leqslant \int_{k-1}^{\infty} x^{-1-\gamma}\,\mathrm{d}x = \gamma^{-1}(k-1)^{-\gamma}$ for all $k \geqslant 2$, hence we get

$$\sum_{i=k}^{\infty} \frac{\alpha_i}{i} \leqslant \frac{c}{\gamma(k-1)^{\gamma}} \quad \text{for all} \quad k \geqslant 2 \tag{2.4}$$

and

$$\sum_{i=1}^{\infty} \frac{\alpha_i}{i} \leqslant c \sum_{i=1}^{\infty} i^{-1-\gamma} = c + c \sum_{i=2}^{\infty} i^{-1-\gamma} \leqslant c + \frac{c}{\gamma(2-1)^{\gamma}} = \frac{c}{\gamma}(\gamma+1).$$

It follows that

$$\beta_1 = \left(\sum_{i=1}^{\infty} \frac{\alpha_i}{i} \right)^{\delta/r} \leqslant \left(\frac{c}{\gamma}(\gamma+1) \right)^{\delta/r} \leqslant \left(\frac{c2^{\gamma}}{\gamma}(\gamma+1) \right)^{\delta/r}. \tag{2.5}$$

On the other hand if $n \geqslant 2$ then (2.4) implies

$$\max_{2 \leqslant k \leqslant n} k^{1/r} \left(\sum_{i=k}^{\infty} \frac{\alpha_i}{i} \right)^{\delta/r} \leqslant \max_{2 \leqslant k \leqslant n} k^{1/r} \left(\frac{c}{\gamma(k-1)^{\gamma}} \right)^{\delta/r}$$
$$\leqslant \max_{2 \leqslant k \leqslant n} \left(\frac{c2^{\gamma}}{\gamma} \right)^{\delta/r} k^{(1-\gamma\delta)/r} = \left(\frac{c2^{\gamma}}{\gamma} \right)^{\delta/r} n^{(1-\gamma\delta)/r}.$$

This inequality, (2.5) and $\lim_{n\to\infty} n^{(1-\gamma\delta)/r} = \infty$ imply for $n \in \mathbb{N}$ large enough

$$\beta_n \leqslant \text{const.} \max \left\{ (\gamma+1)^{\delta/r}, n^{(1-\gamma\delta)/r} \right\} = \text{const.} n^{(1-\gamma\delta)/r}.$$

So $\beta_n n^{-1/r} \leqslant \text{const.} n^{-\gamma\delta/r}$ for $n \in \mathbb{N}$ large enough, which implies the statement. □

3. Mixingales

Let $\{\mathcal{F}_k,\ k \in \mathbb{N}\}$ be a nondecreasing sequence of sub σ-fields of $\mathcal{F}$, $\mathrm{E}_m X_k = \mathrm{E}(X_k \mid \mathcal{F}_m)$ denote the conditional expectation of X_k given $\mathcal{F}_m$ for $m > 0$ and $\mathrm{E}_m X_k = 0$ for $m \leqslant 0$.

Definition 3.1 (McLeish [6], Andrews [1]). The sequence $\{(X_k, \mathcal{F}_k), k \in \mathbb{N}\}$ is an L^r *mixingale* if there exist nonnegative constants $\{c_k, k \geqslant 0\}$ and $\{\psi_k, k \geqslant 0\}$ such that $\psi_k \downarrow 0$ and for all nonnegative integers k and m we have

$$||\, \mathrm{E}_{k-m} X_k||_r \leqslant c_k \psi_m \quad \text{and} \quad ||X_k - \mathrm{E}_{k+m} X_k||_r \leqslant c_k \psi_{m+1},$$

where $||\xi||_r = (\mathrm{E}\,|\xi|^r)^{1/r}$ for any random variable ξ.

Lemma 3.2. *If* $\{(X_k, \mathcal{F}_k), k \in \mathbb{N}\}$ *is* L^r mixingale, *where* $r \geqslant 2$ *and* $\sum_{m=1}^{\infty} \psi_m < \infty$, *then there exists* $c > 0$ *such that for any* $n \in \mathbb{N}$ *and any* $\varepsilon > 0$

$$\mathrm{P}\Big(\max_{k \leqslant n} |S_k| \geqslant \varepsilon\Big) \leqslant c\varepsilon^{-r} \left(\sum_{k=1}^{n} c_k^2\right)^{r/2}.$$

Proof. Hansen [4] proved in Lemma 2 under these conditions, that there exists $c > 0$ such that for any $n \in \mathbb{N}$

$$\mathrm{E}\Big(\max_{k \leqslant n} |S_k|^r\Big) \leqslant c \left(\sum_{k=1}^{n} c_k^2\right)^{r/2}.$$

Hence Markov's inequality implies the statement. □

Theorem 3.3. *Let* $\{(X_k, \mathcal{F}_k), k \in \mathbb{N}\}$ *be an* L^r mixingale, *where* $r \geqslant 2$ *and* $\sum_{m=1}^{\infty} \psi_m < \infty$. *Let* β_n *defined by* (2.1) *with*

$$\alpha_k = \left(\sum_{i=1}^{k} c_i^2\right)^{r/2} - \left(\sum_{i=1}^{k-1} c_i^2\right)^{r/2}.$$

If

$$\sum_{k=1}^{\infty} \frac{c_k^2}{b_k^r} \left(\sum_{i=1}^{k} c_i^2\right)^{r/2-1} < \infty, \tag{3.1}$$

then

$$\lim_{n\to\infty} \frac{\beta_n}{b_n} = 0 \quad \text{and} \quad \frac{S_n}{b_n} = O\left(\frac{\beta_n}{b_n}\right) \quad \text{a.s.}$$

Proof. If $A = \emptyset$ then $\sum_{i \in A} c_i^2 = 0$, hence $\alpha_1 = c_1^r$. Since

$$\sum_{k=1}^{n} \alpha_k = \left(\sum_{i=1}^{n} c_i^2\right)^{r/2},$$

hence Lemma 3.2 implies (2.3). By the mean value theorem

$$x_2^{r/2} - x_1^{r/2} \leqslant (x_2 - x_1)\frac{r}{2} x_2^{r/2-1} \quad \text{for all} \quad 0 \leqslant x_1 \leqslant x_2. \tag{3.2}$$

Using (3.2) with $x_1 = \sum_{i=1}^{k-1} c_i^2$ and $x_2 = \sum_{i=1}^{k} c_i^2$ we get

$$\alpha_k = x_2^{r/2} - x_1^{r/2} \leqslant c_k^2 \frac{r}{2} \left(\sum_{i=1}^{k} c_i^2\right)^{r/2-1}.$$

This inequality and (3.1) imply (2.2). Since every conditions of Theorem 2.3 are satisfied, the statement is proved. □

Lemma 3.4. *If* $\{(X_k, \mathcal{F}_k), k \in \mathbb{N}\}$ *is an* L^r mixingale, *where* $1 < r \leqslant 2$ *and* $\sum_{m=1}^{\infty} \psi_m < \infty$, *then there exists* $c > 0$ *such that for any* $n \in \mathbb{N}$ *and any* $\varepsilon > 0$

$$\mathrm{P}\Big(\max_{k\leqslant n} |S_k| \geqslant \varepsilon\Big) \leqslant c\varepsilon^{-r} \sum_{k=1}^{n} c_k^r.$$

Proof. Hansen [4] proved in Lemma 2 of Erratum under these conditions, that there exists $c > 0$ such that for any $n \in \mathbb{N}$

$$\mathrm{E}\Big(\max_{k\leqslant n} |S_k|^r\Big) \leqslant c \sum_{k=1}^{n} c_k^r.$$

Hence Markov's inequality implies the statement. □

Theorem 3.5. *Let* $\{(X_k, \mathcal{F}_k), k \in \mathbb{N}\}$ *be an* L^r mixingale, *where* $1 < r \leqslant 2$ *and* $\sum_{m=1}^{\infty} \psi_m < \infty$. *Let* β_n *defined by* (2.1) *with* $\alpha_k = c_k^r$. *If*

$$\sum_{k=1}^{\infty} \frac{c_k^r}{b_k^r} < \infty, \tag{3.3}$$

then

$$\lim_{n\to\infty} \frac{\beta_n}{b_n} = 0 \quad \text{and} \quad \frac{S_n}{b_n} = O\left(\frac{\beta_n}{b_n}\right) \quad \text{a.s.}$$

Proof. The statement is a corollary of Lemma 3.4 and Theorem 2.3. □

Corollary 3.6. *Let* $\{(X_k, \mathcal{F}_k), k \in \mathbb{N}\}$ *be an* L^r mixingale, *where* $1 < r \leqslant 2$ *and* $\sum_{m=1}^{\infty} \psi_m < \infty$. *If there exist* $c > 0$ *and* $0 < \gamma < 1$ *such that* $c_k \leqslant ck^{-\gamma/r}$ *for all* $k \in \mathbb{N}$, *then for all* $0 < \delta < 1$

$$\frac{S_n}{n^{1/r}} = O\left(\frac{1}{n^{\gamma\delta/r}}\right) \quad \text{a.s.}$$

Proof. Let $b_k = k^{1/r}$, $\alpha_k = c_k^r$ and $\vartheta(x) = x^{\delta/r}$ $(x \geqslant 0)$, where $0 < \delta < 1$ is a fixed constant. Then for all $k \in \mathbb{N}$

$$\sum_{i=k}^{\infty} \frac{\alpha_i}{i} = \sum_{i=k}^{\infty} \left(\frac{c_i}{b_i}\right)^r \leqslant c^r \sum_{i=k}^{\infty} i^{-1-\gamma}.$$

Hence using Theorem 3.5 and Lemma 2.4 we get

$$\frac{S_n}{n^{1/r}} = O\left(\frac{\beta_n}{n^{1/r}}\right) = O\left(\frac{1}{n^{\gamma\delta/r}}\right) \quad \text{a.s.}$$

□

4. Sequences with superadditive moment function

Definition 4.1 (Móricz [7]). $\{X_k,\ k \in \mathbb{N}\}$ is said to have the r-th $(r > 0)$ *moment function of superadditive structure* if there exists $g\colon \mathbb{N}\cup\{0\}\times\mathbb{N} \to [0,\infty)$ such that

$$g(b,k) + g(b+k,l) \leqslant g(b,k+l) \quad \text{for all} \quad b \in \mathbb{N}\cup\{0\}, k \in \mathbb{N}, l \in \mathbb{N} \tag{4.1}$$

and for some $\alpha > 1$

$$\mathrm{E}\,|S_{b+n} - S_b| \leqslant g^{\alpha}(b,n) \quad \text{for all} \quad b \in \mathbb{N}\cup\{0\}, n \in \mathbb{N}. \tag{4.2}$$

We shall use the notation $g_n = g(0,n)$ $(n \in \mathbb{N})$ and $g_0 = 0$. It is easy to see that $g_n \leqslant g_{n+1}$ for all $n \in \mathbb{N}\cup\{0\}$.

Lemma 4.2. *If $\{X_k,\ k \in \mathbb{N}\}$ has r-th moment function of superadditive structure with $r > 0$, $\alpha > 1$, then there exists a constant $A_{r,\alpha}$ depending only on r and α such that for any $n \in \mathbb{N}$ and any $\varepsilon > 0$*

$$\mathrm{P}\Big(\max_{k\leqslant n} |S_k| \geqslant \varepsilon\Big) \leqslant A_{r,\alpha}\varepsilon^{-r} g_n^{\alpha}.$$

Proof. Móricz proved in [7] under these conditions, that there exists a constant $A_{r,\alpha}$ depending only on r and α, such that for any $n \in \mathbb{N}$

$$\mathrm{E}\Big(\max_{k\leqslant n} |S_k|^r\Big) \leqslant A_{r,\alpha} g_n^{\alpha}.$$

Hence Markov's inequality implies the statement. □

Theorem 4.3. *Assume that $\{X_k,\ k \in \mathbb{N}\}$ has r-th moment function of superadditive structure with $r > 0$, $\alpha > 1$. Let β_n defined by (2.1) with $\alpha_k = g_k^{\alpha} - g_{k-1}^{\alpha}$. If*

$$\sum_{k=1}^{\infty} \frac{g_k^{\alpha} - g_{k-1}^{\alpha}}{b_k^r} < \infty, \tag{4.3}$$

then

$$\lim_{n\to\infty} \frac{\beta_n}{b_n} = 0 \quad \text{and} \quad \frac{S_n}{b_n} = O\left(\frac{\beta_n}{b_n}\right) \quad \text{a.s.}$$

Proof. As g_k increases, we get $\alpha_k \geqslant 0$, thereby (4.3) implies (2.2). On the other hand $\sum_{k=1}^{n} \alpha_k = g_n^{\alpha}$, so Lemma 4.2 implies (2.3). Now applying Theorem 2.3 we get the statement. □

Remark 4.4. Hu and Hu proved Theorem 4.3 in special case $\vartheta(x) = x^{\delta/r}$ $(0 < \delta < 1)$. (See Theorem 2.1 of Hu and Hu [5].)

Corollary 4.5. *Let* $0 < \gamma < 1$*,* $c > 0$*,* $\alpha > 1$ *and* $r > 0$*. If for all* $b \in \mathbb{N}\cup\{0\}, n \in \mathbb{N}$

$$\mathrm{E}\,|S_{b+n} - S_b| \leqslant c\left((b+n)^{(1-\gamma)/\alpha} - b^{(1-\gamma)/\alpha}\right)^{\alpha},$$

then

$$\frac{S_n}{n^{1/r}} = O\left(\frac{1}{n^{\gamma\delta/r}}\right) \quad \text{a.s. for all } \ 0 < \delta < 1.$$

Proof. Let $g\colon \mathbb{N}\cup\{0\}\times\mathbb{N}\cup\{0\} \to [0,\infty)$, $g(i,j) = c^{1/\alpha}\left((i+j)^{(1-\gamma)/\alpha} - i^{(1-\gamma)/\alpha}\right)$. Then (4.1) and (4.2) are satisfied, hence $\{X_k,\ k \in \mathbb{N}\}$ has r-th moment function of superadditive structure.

Now let $b_k = k^{1/r}$ for all $k \in \mathbb{N}$. Since $g_i^\alpha = g^\alpha(0,i) = ci^{1-\gamma}$ for every nonnegative integer i, hence we get

$$\sum_{i=k}^{\infty} \frac{g_i^\alpha - g_{i-1}^\alpha}{b_i^r} = \sum_{i=k}^{\infty} \frac{ci^{1-\gamma} - c(i-1)^{1-\gamma}}{i}$$
$$= c\sum_{i=k}^{\infty} i^{1-\gamma}\left(\frac{1}{i} - \frac{1}{i+1}\right) - c\frac{(k-1)^{1-\gamma}}{k} \leqslant c\sum_{i=k}^{\infty} i^{-1-\gamma}. \tag{4.4}$$

Since (4.4) implies (4.3), hence using Theorem 4.3 we have

$$\frac{S_n}{n^{1/r}} = O\left(\frac{\beta_n}{n^{1/r}}\right) \quad \text{a.s.} \tag{4.5}$$

Let $\vartheta(x) = x^{\delta/r}$, where $0 < \delta < 1$ is a fixed constant. Then (4.4) and Lemma 2.4 imply $\beta_n/n^{1/r} = O\left(1/n^{\gamma\delta/r}\right)$. Hence we get the statement by (4.5). □

Corollary 4.6. *Let* $r > 0$*,* $c > 0$ *and* $1 < \alpha < 2$*. If for all* $b \in \mathbb{N}\cup\{0\}$*,* $n \in \mathbb{N}$

$$\mathrm{E}\,|S_{b+n} - S_b| \leqslant c\left(\sqrt{b+n} - \sqrt{b}\right)^{\alpha},$$

then

$$\frac{S_n}{n^{1/r}} = O\left(n^{-(1-\frac{\alpha}{2})\delta/r}\right) \quad \text{a.s. for all } \ 0 < \delta < 1.$$

Proof. Apply Corollary 4.5 with $\gamma = 1 - \frac{\alpha}{2}$. □

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