

# Some special cases of a general convergence rate theorem in the law of large numbers

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## Abstract

Tómacs in [6] proved a general convergence rate theorem in the law of large numbers for arrays of Banach space valued random elements. We shall study this theorem in case Banach space of type  $\varphi$  and for two special arrays.

**Key Words:** Convergence rates; Arrays of Banach space valued random variables; Banach space of type  $\varphi$

## 1. Introduction and notation

Let  $\mathbb{N}$  be the set of the positive integers and  $\mathbb{R}$  the set of real numbers. Let  $\Phi_0$  denote the set of functions  $f: [0, \infty) \rightarrow [0, \infty)$ , that are nondecreasing. A function  $f \in \Phi_0$  is said to satisfy the  $\Delta_2$ -condition ( $f \sim \Delta_2$ ) if there exists a constant  $c > 0$  such that  $f(2t) \leq cf(t)$  for all  $t > 0$ .

Let  $B$  be a real separable Banach space with norm  $\|\cdot\|$  and zero element  $\mathbf{0}$ . If  $X$  is a  $B$ -valued random variable (r.v.) and  $\mathbf{E}\|X\| < \infty$  then  $\mathbf{E}X$  stands for the Bochner integral of  $X$ .

Throughout the paper let  $\{k_n, n \in \mathbb{N}\}$  be a strictly increasing sequence of positive integers. Let  $\{X_{nk}, n \in \mathbb{N}, k = 1, \dots, k_n\}$  be an array of  $B$ -valued r.v.'s. It is rowwise independent, if  $X_{n1}, \dots, X_{nk_n}$  are independent r.v.'s for any fixed  $n \in \mathbb{N}$ . Let  $S_{k_n} = \sum_{k=1}^{k_n} X_{nk}$ . If  $k_n = n$  for all  $n$ , then we denote  $S_{k_n}$  by  $S_n$ . This corresponds to the case of ordinary sequences.

The array  $\{X_{nk}, n \in \mathbb{N}, k = 1, \dots, k_n\}$  is said to be *bounded in probability* if for all  $\varepsilon > 0$  there exists  $A > 0$  such that  $\mathbf{P}(\|X_{nk}\| \geq A) < \varepsilon$  for all  $n \in \mathbb{N}$  and  $k = 1, \dots, k_n$ .

The following remark give a sufficient condition for the boundedness in probability.

**Remark 1.1.** If there exists a constant  $M > 0$  such that  $\mathbf{E} \|X_{nk}\| \leq M$  for every  $n \in \mathbb{N}, k = 1, \dots, k_n$ , then the array  $\{X_{nk}, n \in \mathbb{N}, k = 1, \dots, k_n\}$  is bounded in probability. (The reader can readily verify this statement.)

**Definition 1.2** (Gut [2]). We say that the array  $\{X_{nk}, n \in \mathbb{N}, k = 1, \dots, k_n\}$  is *weakly mean dominated* (w.m.d.) by the r.v.  $X$ , if for some  $\gamma > 0$ ,

$$\frac{1}{k_n} \sum_{k=1}^{k_n} \mathbf{P}(\|X_{nk}\| > t) \leq \gamma \mathbf{P}(|X| > t) \quad \text{for all } t \geq 0 \quad \text{and } n \in \mathbb{N}.$$

The following theorem a general convergence rate theorem, which is proved in [6].

**Theorem 1.3** (Tórnács [6], Theorem 3.1). *Let  $\{X_{nk}, n \in \mathbb{N}, k = 1, \dots, n\}$  be an array of rowwise independent  $B$ -valued r.v.'s which is w.m.d. by the r.v.  $X$ . Assume that there exists a sequence  $\{\gamma_n, n \in \mathbb{N}\}$  of positive real numbers such that  $\{\|S_n\|/\gamma_n, n \in \mathbb{N}\}$  is bounded in probability. Let  $\alpha, \vartheta, \varphi \in \Phi_0$ , and assume that  $\alpha$  is not bounded,  $\vartheta, \varphi \sim \Delta_2$ ,  $\vartheta \neq 0$ . Let  $\beta(n) = \varphi(\alpha(n+1)) - \varphi(\alpha(n))$ ,  $n = 0, 1, 2, \dots$ . Assume that  $\mathbf{E} \varphi(|X|) < \infty$ ,  $\mathbf{E} \vartheta(|X|) < \infty$  and  $\lim_{n \rightarrow \infty} \alpha(n)/\gamma_n = \infty$ .*

*Let either  $\mu(n) = \beta(n-1)$  for all  $n \in \mathbb{N}$  or  $\mu(n) = \beta(n)$  for all  $n \in \mathbb{N}$ . In second case assume that there exists a constant  $c > 0$  such that for  $n \in \mathbb{N}$  large enough  $c\beta(n) \leq \beta(n-1)$ .*

*Let  $n_0 \in \mathbb{N}$  be such that  $\vartheta(\alpha(n)) > 0$  for all  $n \geq n_0$ . If there exist  $j \in \mathbb{N}$  and  $r > 0$  such that*

$$\sum_{n=n_0}^{\infty} \frac{\mu(n)}{n} \left( \frac{rn + \vartheta(\gamma_n)}{\vartheta(\alpha(n))} \right)^{2^j} < \infty,$$

then

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \mathbf{P}(\|S_n\| > \varepsilon \alpha(n)) < \infty \quad \text{for all } \varepsilon > 0.$$

In the following two corollaries of Theorem 1.3 we use some special notations: Following Gut [1], introduce the functions  $\psi$  and  $M_r$  with

$$\psi(t) = \text{Card}\{n \in \mathbb{N} : k_n \leq t\} \quad \text{for } t \geq 0,$$

and

$$M_r(t) = \sum_{i=1}^{[t]} k_i^{r-1} \quad \text{if } t \geq 1 \quad \text{and} \quad M_r(t) = k_1^{r-1} \quad \text{if } 0 \leq t < 1,$$

where  $r \in \mathbb{R}$ ,  $\text{Card}A$  is the cardinality of the set  $A$  and  $[.]$  denotes the integer function. Let  $M = M_2$ . Let  $f \circ g$  be the composite function of functions  $f$  and  $g$ .

**Remark 1.4.**  $M_r \circ \psi \in \Phi_0$  and

$$(M_r \circ \psi)(t) = M_r(\psi(t)) = \begin{cases} \sum_{i=1}^n k_i^{r-1} = M_r(n), & \text{if } k_n \leq t < k_{n+1}, \\ k_1^{r-1} = M_r(1), & \text{if } 0 \leq t < k_1. \end{cases}$$

The following corollary is a generalization of Theorem 6.2 of Fazekas [5].

**Corollary 1.5** (Tómacs [6], Corollary 3.2). *Let  $\{X_{nk}, n \in \mathbb{N}, k = 1, \dots, k_n\}$  be an array of rowwise independent  $B$ -valued r.v.'s which is w.m.d. by the r.v.  $X$ . Let  $M \circ \psi \sim \Delta_2$ ,  $r, s, t > 0$ ,  $rs > t$ . Assume that  $\{\|S_{k_n}\|/k_n^{1/s}, n \in \mathbb{N}\}$  is bounded in probability. Furthermore, if  $r > 2$  we assume that  $\{M(n)/M(n-1), n \in \mathbb{N}\}$  is bounded. If  $\mathbf{E}M^{r/2}(\psi(|X|^{t/r})) < \infty$  and  $\mathbf{E}|X|^s < \infty$ , then*

$$\sum_{n=1}^{\infty} (M(n))^{r/2-1} \mathbf{P} \left( \|S_{k_n}\| > \varepsilon k_n^{r/t} \right) < \infty \quad \text{for all } \varepsilon > 0.$$

The following corollary is a version of Corollary 4.1 of Hu et al. [3].

**Corollary 1.6** (Tómacs [6], Corollary 3.3). *Let  $\{X_{nk}, n \in \mathbb{N}, k = 1, \dots, k_n\}$  be an array of rowwise independent  $B$ -valued r.v.'s which is w.m.d. by the r.v.  $X$ . Let  $r \in \mathbb{R}$ ,  $0 < t < s$  and  $M_r \circ \psi \sim \Delta_2$ . Assume that  $\{\|S_{k_n}\|/k_n^{1/s}, n \in \mathbb{N}\}$  is bounded in probability. If  $\mathbf{E}M_r(\psi(|X|^t)) < \infty$  and  $\mathbf{E}|X|^s < \infty$ , then*

$$\sum_{n=1}^{\infty} k_n^{r-2} \mathbf{P} \left( \|S_{k_n}\| > \varepsilon k_n^{1/t} \right) < \infty \quad \text{for all } \varepsilon > 0.$$

In Section 2 we give a sufficient condition for the boundedness in probability and in Section 3 we study two concrete sequences  $k_n$  in Corollary 1.5 and 1.6.

## 2. The boundedness in probability in case Banach space of type $\varphi$

If  $B$  has an appropriate geometric property, then a moment condition can imply the boundedness of  $\{\|S_{k_n}\|/\gamma_{k_n}, n \in \mathbb{N}\}$ .

**Definition 2.1.** A function  $\varphi: [0, \infty) \rightarrow [0, \infty)$  is said to be an *Orlicz function* if it is continuous, convex,  $\varphi(0) = 0$ ,  $\varphi(t) > 0$  for  $t > 0$  and  $\lim_{t \rightarrow \infty} \varphi(t) = \infty$ . For an Orlicz function  $\varphi$  the *Orlicz space*  $l_\varphi(B)$  consists of those  $B$ -valued sequences  $\{u_n, n \in \mathbb{N}\}$  for which

$$\sum_{n=1}^{\infty} \varphi(\|u_n\|/a) < \infty \quad \text{for some } a > 0.$$

Let  $\varepsilon_1, \varepsilon_2, \dots$  be independent r.v.'s with  $\mathbf{P}(\varepsilon_n = 1) = \mathbf{P}(\varepsilon_n = -1) = 1/2$  for all  $n \in \mathbb{N}$ .  $B$  is said to be of *type  $\varphi$* , if  $\sum_{n=1}^{\infty} \varepsilon_n u_n$  converges in probability for all  $\{u_n, n \in \mathbb{N}\} \in l_\varphi(B)$ .

**Definition 2.2.** An Orlicz function  $\varphi$  is said to satisfy the  $\Delta_2^0$ -condition ( $\varphi \sim \Delta_2^0$ ) if there exist constants  $c > 0$  and  $t_0 > 0$  such that  $\varphi(2t) \leq c\varphi(t)$  is satisfied for all  $0 \leq t \leq t_0$ .

**Lemma 2.3.** Let  $\varphi$  be an Orlicz function and  $\varphi \sim \Delta_2^0$ .  $B$  is of type  $\varphi$  iff there exists a constant  $c > 0$  such that

$$\mathbf{E} \left\| \sum_{k=1}^n X_k \right\| \leq c \mathbf{E} \inf_{y>0} \left\{ \frac{1}{y} \left( 1 + \sum_{k=1}^n \varphi(y \|X_k\|) \right) \right\}$$

for all  $n \in \mathbb{N}$  and every independent  $B$ -valued r.v.'s  $X_1, \dots, X_n$  with  $\mathbf{E}X_k = \mathbf{0}$ ,  $k = 1, \dots, n$ .

For the proof see Fazekas [4].

The following lemma is a generalization of Lemma 2.1 of Gut [2] and Lemma 2.7 (b) of Fazekas [5].

**Lemma 2.4** (Tómacs [6], Lemma 4.4). Let  $\{X_{nk}, n \in \mathbb{N}, k = 1, \dots, k_n\}$  be an array of  $B$ -valued r.v.'s which is w.m.d. by the r.v.  $X$  and constant  $\gamma$ . If  $\varphi \in \Phi_0$  then

$$\frac{1}{k_n} \sum_{k=1}^{k_n} \mathbf{E} \varphi(\|X_{nk}\|) \leq \max\{1, \gamma\} \mathbf{E} \varphi(|X|).$$

The following theorem shows that in Theorem 1.3 we can write moment conditions instead of the boundedness of  $\{\|S_{k_n}\|/\gamma_{k_n}, n \in \mathbb{N}\}$  if  $B$  is of type  $\varphi$ .

**Theorem 2.5.** Let  $\varphi \in \Phi_0$  be a submultiplicative Orlicz function,  $\varphi \sim \Delta_2^0$  and let  $B$  be a space of type  $\varphi$ . Let  $\{X_{nk}, n \in \mathbb{N}, k = 1, \dots, k_n\}$  be an array of rowwise independent  $B$ -valued r.v.'s which is w.m.d. by the r.v.  $X$ . Assume that the sequence  $\{k_n \varphi(1/\gamma_{k_n}), n \in \mathbb{N}\}$  is bounded for some sequence  $\{\gamma_n, n \in \mathbb{N}\}$  of positive real numbers. If  $\mathbf{E}X_{nk} = \mathbf{0}$  for every  $n \in \mathbb{N}$ ,  $k = 1, \dots, k_n$  and  $\mathbf{E} \varphi(|X|) < \infty$ , then  $\{\|S_{k_n}\|/\gamma_{k_n}, n \in \mathbb{N}\}$  is bounded in probability.

**Proof.** By Lemma 2.3 and 2.4 there exists a constant  $c > 0$  such that

$$\begin{aligned} \frac{\mathbf{E} \|S_{k_n}\|}{\gamma_{k_n}} &\leq \frac{c}{\gamma_{k_n}} \mathbf{E} \inf_{y>0} \left\{ \frac{1}{y} \left( 1 + \sum_{k=1}^{k_n} \varphi(y \|X_{nk}\|) \right) \right\} \\ &\leq c \mathbf{E} \left( 1 + \sum_{k=1}^{k_n} \varphi(\|X_{nk}\|/\gamma_{k_n}) \right) \\ &\leq c(1 + \varphi(1/\gamma_{k_n}) \max\{1, \gamma\} k_n \mathbf{E} \varphi(|X|)). \end{aligned}$$

Thus Remark 1.1 implies the statement.  $\square$

### 3. Convergence rate theorems for two concrete sequences $k_n$

**Lemma 3.1.**  $f \sim \Delta_2$  iff there exist constants  $k > 1$  and  $c > 0$  such that

$$f(kt) \leq cf(t) \quad \text{for all } t > 0. \tag{3.1}$$

**Proof.** If  $f \sim \Delta_2$  then in case  $k = 2$  we get (3.1). Now suppose that there exist constants  $k > 1$  and  $c > 0$  such that the inequality (3.1) is true for all  $t > 0$ . Then we can obtain with induction that

$$f(k^n t) \leq c^n f(t) \quad \text{for all } t > 0 \quad \text{and for all } n \in \mathbb{N}.$$

It follows that there exists  $n_0 \in \mathbb{N}$  such that

$$f(2t) \leq f(k^{n_0} t) \leq c^{n_0} f(t) \quad \text{for all } t > 0.$$

Thus we get  $f \sim \Delta_2$ . □

The reader can readily verify the following lemma.

**Lemma 3.2.** Let  $g: [k_1, \infty) \rightarrow \mathbb{R}$  be a nondecreasing function which has the property that  $g(k_n) \geq M_r(n)$  for all  $n \in \mathbb{N}$ . Then  $M_r(\psi(x)) \leq g(x)$  for all  $x \geq k_1$ .

**Lemma 3.3.** Let  $r \in \mathbb{R}$ . Assume that there exists strictly increasing sequence  $\{a_n, n \in \mathbb{N}\}$  of positive integers and there exist constants  $k > 1, c > 0$  such that

$$\frac{k_n}{k_{a_n}} \leq \frac{1}{k} \quad \text{and} \quad \frac{M_r(a_n)}{M_r(n-1)} \leq c \quad \text{for all } n \in \mathbb{N}.$$

Then  $M_r \circ \psi \sim \Delta_2$ .

**Proof.** Assume that  $k_n \leq t < k_{n+1}$ . Then Remark 1.4 implies

$$M_r(\psi(kt)) \leq M_r(\psi(kk_{n+1})) \leq M_r(\psi(k_{a_{n+1}})) = M_r(a_{n+1}) \leq cM_r(n) = cM_r(\psi(t)).$$

Similarly if  $0 < t < k_1$  then

$$M_r(\psi(kt)) \leq M_r(\psi(kk_1)) \leq M_r(\psi(k_{a_1})) = M_r(a_1) \leq cM_r(0) = cM_r(\psi(t)).$$

It follows that  $M_r(\psi(kt)) \leq cM_r(\psi(t))$  for all  $t > 0$ . Thus, by Lemma 3.1 we get the statement. □

**Lemma 3.4.** Let  $l \in \mathbb{N}$ . Then

$$\lim_{n \rightarrow \infty} \frac{1^k + 2^k + \dots + (ln)^k}{1^k + 2^k + \dots + (n-1)^k} = \begin{cases} l^{k+1}, & \text{if } k > -1, \\ 1, & \text{if } k \leq -1. \end{cases}$$

**Proof.** It is easy to see that

$$x^{k+1} - (x-1)^{k+1} \leq (k+1)x^k \leq (x+1)^{k+1} - x^{k+1} \quad \text{for all } x \geq 1, k \geq 0$$

and

$$(x+1)^{k+1} - x^{k+1} \leq (k+1)x^k \leq x^{k+1} - (x-1)^{k+1} \quad \text{for all } x \geq 1, -1 < k < 0.$$

Apply these inequalities for  $x = 1, 2, \dots, n$ . Then we have

$$\lim_{n \rightarrow \infty} \frac{1^k + 2^k + \dots + n^k}{n^{k+1}} = \frac{1}{k+1} \quad \text{for all } k > -1,$$

which implies the statement for  $k > -1$ .

It is well known that  $\frac{1}{1^c} + \frac{1}{2^c} + \dots + \frac{1}{n^c}$  is convergent if  $c > 1$ . It follows that the statement is true in case  $k < -1$  as well.

Finally in case  $k = -1$  the inequalities

$$1 + \frac{\frac{l-1}{l}}{1 + \frac{1}{2} + \dots + \frac{1}{n-1}} < \frac{1 + \frac{1}{2} + \dots + \frac{1}{ln}}{1 + \frac{1}{2} + \dots + \frac{1}{n-1}} < 1 + \frac{l}{1 + \frac{1}{2} + \dots + \frac{1}{n-1}}$$

imply the statement. □

**Lemma 3.5.** *Let  $k_1, d \in \mathbb{N}$ ,  $q \in \mathbb{N} \setminus \{1\}$ . If  $k_n = k_1 q^{n-1}$  or  $k_n = k_1 n^d$  then  $M_r \circ \psi \sim \Delta_2$  for all  $r \in \mathbb{R}$ .*

**Proof.** In the first case, when  $k_n = k_1 q^{n-1}$ , let  $a_n = n+1$  and  $k = q$ . Then

$$\frac{k_n}{k_{a_n}} = \frac{k_1 q^{n-1}}{k_1 q^n} = \frac{1}{q} \leq \frac{1}{k}.$$

Let  $Q = q^{r-1}$  and assume that  $r > 1$ . In this case  $|1/Q| < 1$ , thus we get

$$\frac{M_r(a_n)}{M_r(n-1)} = \frac{M_r(n+1)}{M_r(n-1)} = \frac{1 + Q + \dots + Q^n}{1 + Q + \dots + Q^{n-2}} = \frac{Q^2 - \frac{1}{Q^{n-1}}}{1 - \frac{1}{Q^{n-1}}} \rightarrow Q^2.$$

If  $r < 1$  then  $1/Q > 1$ , thus

$$\frac{M_r(a_n)}{M_r(n-1)} = \frac{Q^2 - \frac{1}{Q^{n-1}}}{1 - \frac{1}{Q^{n-1}}} \rightarrow 1.$$

If  $r = 1$  then  $Q = 1$ , so

$$\frac{M_r(a_n)}{M_r(n-1)} = \frac{n+1}{n-1} \rightarrow 1.$$

Thus we get that  $\frac{M_r(a_n)}{M_r(n-1)}$  is bounded for all  $r \in \mathbb{R}$ . Hence conditions of Lemma 3.3 are satisfied, which implies the statement.

In the second case, when  $k_n = k_1 n^d$ , let  $a_n = 2n$  and  $k = 2^d$ . Then

$$\frac{k_n}{k_{a_n}} = \frac{k_1 n^d}{k_1 (2n)^d} = \frac{1}{2^d} \leq \frac{1}{k}.$$

On the other hand it follows from Lemma 3.4 that  $\frac{M_r(a_n)}{M_r(n-1)}$  is bounded for all  $r \in \mathbb{R}$ . So Lemma 3.3 implies the statement.  $\square$

**Theorem 3.6.** *Let  $\{X_{nk}, n \in \mathbb{N}, k = 1, \dots, k_1 n^d\}$  ( $k_1, d \in \mathbb{N}$  are fixed) be an array of rowwise independent  $B$ -valued r.v.'s which is w.m.d. by the r.v.  $X$ . Let  $t > 0$ ,  $r \geq 2d/(d + 1)$ ,  $s > t/r$  and  $v = \max\{s, t(d + 1)/(2d)\}$ . If  $\{\|S_{k_1 n^d}\|/n^{d/s}, n \in \mathbb{N}\}$  is bounded in probability and  $\mathbf{E}|X|^v < \infty$ , then*

$$\sum_{n=1}^{\infty} n^{(d+1)(r/2-1)} \mathbf{P}\left(\|S_{k_1 n^d}\| > \varepsilon n^{dr/t}\right) < \infty \quad \text{for all } \varepsilon > 0.$$

**Proof.** We shall prove that conditions of Corollary 1.5 are satisfied. Let  $k_n = k_1 n^d$ . Then by Lemma 3.4  $\{M(n)/M(n - 1), n \in \mathbb{N}\}$  is bounded. Let  $Y = M^{r/2}(\psi(|X|^{t/r}))$ . Now we turn to the proof of  $\mathbf{E}Y < \infty$ . It is well known that

$$1^d + \dots + n^d = a_1 n^{d+1} + a_2 n^d + \dots + a_{d+2}$$

for some  $a_1, a_2, \dots, a_{d+2} \in \mathbb{R}$ . Let

$$g: [k_1, \infty) \rightarrow \mathbb{R}, \quad g(x) = \sum_{i=1}^{d+2} |a_i| (k_1^{i-2} x^{d+2-i})^{1/d}.$$

Then  $g$  is nondecreasing,  $g(k_n) \geq M(n)$  and  $g(x) \leq \text{const.}x^{(d+1)/d}$  for all  $x \geq k_1$ . Therefore by Lemma 3.2 we have

$$M^{r/2}\left(\psi(x^{t/r})\right) \leq \text{const.}x^{t(d+1)/(2d)} \quad \text{for all } x^{t/r} \geq k_1.$$

It follows that

$$Y = Y\mathbf{I}(|X|^{t/r} < k_1) + Y\mathbf{I}(|X|^{t/r} \geq k_1) \leq k_1^{r/2} + \text{const.}|X|^{t(d+1)/(2d)},$$

where  $\mathbf{I}(A)$  denotes the indicator function of the set  $A$ . So  $\mathbf{E}Y < \infty$ . By Lemma 3.5  $M \circ \psi \sim \Delta_2$ . It is easy to see that the other conditions of Corollary 1.5 hold true as well, on the other hand  $M(n) \geq \text{const.}n^{d+1}$ . So this theorem is consequence of Corollary 1.5.  $\square$

**Theorem 3.7.** *Let  $\{X_{nk}, n \in \mathbb{N}, k = 1, \dots, k_1 q^{n-1}\}$  ( $k_1 \in \mathbb{N}, q \in \mathbb{N} \setminus \{1\}$  are fixed) be an array of rowwise independent  $B$ -valued r.v.'s which is w.m.d. by the r.v.  $X$ . Let  $w \geq 0$ ,  $t > 0$ ,  $s > t$  and  $v = \max\{s, t(w + 1)\}$ . If  $\{\|S_{k_1 q^{n-1}}\|/q^{n/s}, n \in \mathbb{N}\}$  is bounded in probability and  $\mathbf{E}|X|^v < \infty$ , then*

$$\sum_{n=1}^{\infty} q^{nw} \mathbf{P}\left(\|S_{k_1 q^{n-1}}\| > \varepsilon q^{n/t}\right) < \infty \quad \text{for all } \varepsilon > 0.$$

**Proof.** We shall prove that conditions of Corollary 1.6 are satisfied. Let  $k_n = k_1 q^{n-1}$ ,  $r = w + 2$  and  $Y = M_r(\psi(|X|^t))$ . Then  $M_r(n) = k_1^{r-1} \frac{Q^n - 1}{Q - 1}$ , where  $Q = q^{r-1}$ . Let

$$g: [k_1, \infty) \rightarrow \mathbb{R}, \quad g(x) = k_1^{r-1} \frac{Q^{1+\log(x/k_1)/\log q} - 1}{Q - 1}.$$

Then  $g$  is nondecreasing,  $g(k_n) = M_r(n)$  and  $g(x) \leq \text{const.}x^{r-1}$  for all  $x \geq k_1$ . Therefore by Lemma 3.2 we have

$$M_r(\psi(x^t)) \leq \text{const.}x^{t(w+1)} \quad \text{for all } x^t \geq k_1.$$

It follows that

$$Y = YI(|X|^t < k_1) + YI(|X|^t \geq k_1) \leq k_1^{r-1} + \text{const.}|X|^{t(w+1)}.$$

So  $EY < \infty$ . By Lemma 3.5  $M_r \circ \psi \sim \Delta_2$ . The other conditions of Corollary 1.6 hold true as well. Thus Corollary 1.6 implies the statement.  $\square$

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