Convergence rates in the law of large numbers for arrays of Banach space valued random elements

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Received 23 February 2004; received in revised form 6 December 2004

Abstract

A general convergence rate theorem is obtained for arrays of Banach space valued random elements. This theorem gives a unified approach to prove and extend several known results.

Keywords: Convergence rates; Complete convergence; Arrays of Banach space valued random variables

1. Introduction

Several papers are devoted to the study of convergence rates in the law of large numbers. The well-known theorem of Baum and Katz (1965) states the following. Let \( X_1, X_2, \ldots \) be independent identically distributed random variables with \( E X_k = 0 \) if \( E|X_k| < \infty \). Let \( t > 0 \), \( r \geq 1 \) and \( 2r > t \). Then \( E|X_k|^r < \infty \) if and only if

\[
\sum_{n=1}^{\infty} n^{r-2} P(|S_n| > \varepsilon n^{r/2}) < \infty \quad \text{for all} \quad \varepsilon > 0.
\]

Earlier versions of this theorem were obtained by Hsu and Robbins (1947), Erdős (1949,1950) and Spitzer (1956). The result was extended to Banach space valued random variables (Jain, 1975; Woyczyński, 1980), to arrays of random variables (Hu et al., 1989; Gut, 1992). For the recent progress in this field see Ahmed et al. (2002) and Csörgő (2003).

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Instead of the whole sequence \( S_n \), one can study the subsequence \( S_{k_n} \). Most of the papers study subsequences by methods that are different from the ones used for the whole sequence. Similarly, papers dealing with arrays of random variables (see Gut, 1992; Fazekas, 1992; Hu et al., 1999) offer their own method to handle general arrays. The aim of this note is to show that an appropriate version of the classic result of Jain (1975) on \( S_n \) (Theorem 3.3) implies theorems on \( S_{k_n} \) for a broad class of \( k_n \).

Throughout the paper we study Banach space valued random variables. However, some of our results are new for real variables, too. In Section 2 we introduce notation. The main results are in Section 3. Theorem 3.1 is a generalization of Theorem 3.3 of Jain (1975). The idea in Theorem 3.1 is the following. When we apply Hoffmann–Jørgensen’s inequality, we use two different functions to obtain upper bounds for the two terms in the inequality. The theorem obtained seems to be difficult, but when we choose appropriate weight functions we can obtain several known theorems for general arrays like \( X_{n1}, \ldots, X_{n_k} \). Corollaries 3.2 and 3.3 are versions of Theorem 6.2 of Fazekas (1992) and Corollary 4.1 of Hu et al. (1999), respectively. In Section 4 we give the proofs.

In Section 5 we specialize our result for Banach spaces which are of type \( p \). Then we obtain new proofs for results in Fazekas (1992) and Hu et al. (1999).

2. Notation

Let \( \mathbb{N} \) be the set of the positive integers, \( \mathbb{R} \) the set of real numbers, \( a \lor b = \max\{a, b\} \) and \( a \land b = \min\{a, b\} \), where \( a, b \in \mathbb{R} \). Denote by \( R_f \) the range of the function \( f \) and by \( f \circ g \) the composite function of functions \( f \) and \( g \).

Let \( \Phi_0 \) denote the set of functions \( f: [0, \infty) \to [0, \infty) \), that are nondecreasing. A function \( f \in \Phi_0 \) is said to satisfy the \( A_2 \)-condition \( (f \sim A_2) \) if there exists a constant \( c > 0 \), such that \( f(2t) \leq cf(t) \) for all \( t > 0 \). It is clear that \( f \sim A_2 \) iff for every fixed \( k > 1 \), there exists a constant \( c > 1 \), such that \( f(kt) \leq cf(t) \) for all \( t > 0 \).

Throughout the paper let \( \{k_n, n \in \mathbb{N}\} \) be a strictly increasing sequence of positive integers. Following Gut (1985), introduce the functions \( \psi \) and \( M_r \) with

\[
\psi(t) = \text{Card}\{n \in \mathbb{N}: k_n \leq t\} \quad \text{for} \quad t > 0 \quad \text{and} \quad \psi(0) = 0,
\]

and

\[
M_r(t) = \sum_{i=1}^{[t]} k_i^{r-1} \quad \text{if} \quad t \geq 1 \quad \text{and} \quad M_r(t) = k_1^{r-1} \quad \text{if} \quad 0 \leq t < 1,
\]

where \( r \in \mathbb{R} \), \( \text{Card} A \) is the cardinality of the set \( A \) and \( [\cdot] \) denotes the integer function. Let \( M = M_2 \).

Let \( B \) be a real separable Banach space with norm \( \| \cdot \| \) and zero element \( 0 \). If \( X \) is a \( B \)-valued random variable (r.v.) and \( E\|X\| < \infty \) then \( EX \) stands for the Bochner integral of \( X \).

\( X \) is symmetric if \( X \) and \( -X \) have same distribution. The symmetrization procedure consists in assigning to the r.v. \( X \) the symmetrized r.v. \( X^* = X - X' \), where \( X' \) is independent of \( X \) and has the same distribution. Then
\[ P(\|X\| < t)P(\|X\| > 2t) \leq P(\|X^*\| > t) \leq 2P(\|X - b\| > t/2) \]  
(2.1)

for all \( t \geq 0 \) and \( b \in B \).

Let \( \{X_{nk}, n \in \mathbb{N}, k = 1, \ldots, k_n\} \) be an array of \( B \)-valued r.v.’s. It is rowwise independent, if \( X_{n1}, \ldots, X_{nk_n} \) are independent r.v.’s for any fixed \( n \in \mathbb{N} \). Let \( S_{kn} = \sum_{k=1}^{k_n} X_{nk} \). If \( k_n = n \) for all \( n \), then we denote \( S_{kn} \) by \( S_n \). This corresponds to the case of ordinary sequences.

**Definition 2.1** (Gut, 1992). We say that the array \( \{X_{nk}, n \in \mathbb{N}, k = 1, \ldots, k_n\} \) is weakly mean dominated (w.m.d.) by the r.v. \( X \), if for some \( \gamma > 0 \),

\[
\frac{1}{k_n} \sum_{k=1}^{k_n} P(\|X_{nk}\| > t) \leq \gamma P(|X| > t) \quad \text{for all} \quad t \geq 0 \text{ and } n \in \mathbb{N}.
\]  
(2.2)

**Remark 2.2.** If \( \{X_{nk}, n \in \mathbb{N}, k = 1, \ldots, k_n\} \) is w.m.d. by the r.v. \( X \), then

\[
\frac{1}{k_n} \sum_{k=1}^{k_n} P(\|X_{nk}\| \geq t) \leq \gamma P(|X| \geq t) \quad \text{for all} \quad t > 0 \text{ and } n \in \mathbb{N}.
\]  
(2.3)

3. A general convergence rate theorem

Our main result is Theorem 3.1. It concerns the case of \( k_n \equiv n \). However, its general setup allows us to apply it for general sequences \( k_n \) (see Corollaries 3.2, 3.3, 5.2 and Theorem 5.5).

**Theorem 3.1.** Let \( \{X_{nk}, n \in \mathbb{N}, k = 1, \ldots, n\} \) be an array of rowwise independent \( B \)-valued r.v.’s which is w.m.d. by the r.v. \( X \). Assume that there exists a sequence \( \{\gamma_n, n \in \mathbb{N}\} \) of positive real numbers such that \( \{\|S_n\|/\gamma_n, n \in \mathbb{N}\} \) is bounded in probability. Let \( \alpha, \beta, \phi \in \Phi_0 \), and assume that \( \alpha \) is not bounded, \( \beta, \phi \sim A_2, \beta \neq 0 \). Let

\[
\beta(n) = \alpha(n + 1) - \alpha(n), \quad n = 0, 1, 2, \ldots.
\]

Assume that

\[
E \phi(|X|) < \infty, \quad E \beta(|X|) < \infty \quad \text{and} \quad \lim_{n \to \infty} \frac{\alpha(n)}{\gamma_n} = \infty.
\]

Let either

\[
\mu(n) = \beta(n - 1) \quad \text{for all} \quad n \in \mathbb{N}
\]  
(3.1)

or

\[
\mu(n) = \beta(n) \quad \text{for all} \quad n \in \mathbb{N}.
\]  
(3.2)

In case (3.2) assume that there exists a constant \( c > 0 \) such that for \( n \in \mathbb{N} \) large enough

\[
c \beta(n) \leq \beta(n - 1).
\]  
(3.3)

Let \( n_0 \in \mathbb{N} \) be such that \( \beta(\alpha(n)) > 0 \) for all \( n \geq n_0 \). If there exist \( j \in \mathbb{N} \) and \( r > 0 \) such that

\[
\sum_{n=n_0}^{\infty} \frac{\mu(n)}{n} \left( \frac{rn + \beta(\gamma_n)}{\beta(\alpha(n))} \right)^2 < \infty,
\]  
(3.4)

then
\[
\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \mathbb{P}(\|S_n\| > \varepsilon n) < \infty \quad \text{for all} \quad \varepsilon > 0.
\]
(3.5)

The following corollary is a generalization of Theorem 6.2 of Fazekas (1992).

**Corollary 3.2.** Let \( \{X_{nk}, n \in \mathbb{N}, k = 1, \ldots, k_n\} \) be an array of rowwise independent \( B \)-valued r.v.’s which is w.m.d. by the r.v. \( X \). Let \( M \circ \psi \sim \Delta_2 \), \( r, s, t > 0, rs > t \). Assume that \( \{\|S_{kn}\|/k_n^{1/s}, n \in \mathbb{N}\} \) is bounded in probability. Furthermore, if \( r > 2 \) we assume that \( \{M(n)/M(n-1), n \in \mathbb{N}\} \) is bounded. If

\[
E M^{r/2}(\psi(|X|^r)) < \infty \quad \text{and} \quad E|X|^s < \infty,
\]
then

\[
\sum_{n=1}^{\infty} (M(n))^{r/2-1}\mathbb{P}(\|S_{kn}\| > \varepsilon k_n^{r/l}) < \infty \quad \text{for all} \quad \varepsilon > 0.
\]

The following corollary is a version of Corollary 4.1 of Hu et al. (1999).

**Corollary 3.3.** Let \( \{X_{nk}, n \in \mathbb{N}, k = 1, \ldots, k_n\} \) be an array of rowwise independent \( B \)-valued r.v.’s which is w.m.d. by the r.v. \( X \). Let \( r \in \mathbb{R}, 0 < t < s \) and \( M_r \circ \psi \sim \Delta_2 \). Assume that \( \{\|S_{kn}\|/k_n^{1/s}, n \in \mathbb{N}\} \) is bounded in probability. If

\[
E M_r(\psi(|X|^t)) < \infty \quad \text{and} \quad E|X|^s < \infty,
\]
then

\[
\sum_{n=1}^{\infty} k_n^{r-2}P(\|S_{kn}\| > \varepsilon k_n^{1/l}) < \infty \quad \text{for all} \quad \varepsilon > 0.
\]

4. Proofs

We start with some preliminary results. The following lemma is a version of Lemma 2.2 of Jain (1975).

**Lemma 4.1.** Let \( X \) be a r.v., \( \varphi, \psi \in \Phi_0, \beta(n) = \varphi(\psi(n+1)) - \varphi(\psi(n)), n = 0, 1, 2, \ldots \). If \( E \varphi(|X|) < \infty \), then

\[
\sum_{n=1}^{\infty} \beta(n-1)\mathbb{P}(|X| > \varphi(n)) < \infty.
\]

**Proof.** With notation \( \Theta_n = \varphi(\psi(n)) \) we have

\[
E \varphi(|X|) \geq \sum_{i=1}^{\infty} \Theta_i P(\Theta_i < \varphi(|X|) < \Theta_{i+1}) \geq \sum_{i=1}^{\infty} \sum_{n=1}^{i} \beta(n-1)P(\Theta_i < \varphi(|X|) < \Theta_{i+1})
\]

\[
= \sum_{n=1}^{\infty} \beta(n-1) \sum_{i=n}^{\infty} P(\Theta_i < \varphi(|X|) < \Theta_{i+1}) \geq \sum_{n=1}^{\infty} \beta(n-1)\mathbb{P}(|X| > \varphi(n)). \quad \Box
\]

The following lemma is due to Hoffmann–Jørgensen (1974) and Jain (1975).
Lemma 4.2. Let \( X_1, \ldots, X_n \) be \( B \)-valued, independent, symmetric r.v.’s and \( j \in \mathbb{N} \). Then there exists \( A_j, B_j \geq 0 \), depending only on \( j \), such that

\[
P \left( \left\| \sum_{k=1}^{n} X_k \right\| > \frac{3}{2} t \right) \leq 2A_j P \left( \max_{1 \leq k \leq n} \left\| X_k \right\| > t \right) + 2B_j P^2 \left( \left\| \sum_{k=1}^{n} X_k \right\| > t \right)
\]

for all \( t > 0 \). (\( A_1 = 1, B_1 = 4 \))

The following lemma is a generalization of Theorem 3.1 of Jain (1975) and Lemma 2.6 of Fazekas (1992).

Lemma 4.3. Let \( \{ X_{nk}, n \in \mathbb{N}, k = 1, \ldots, k_n \} \) be an array of rowwise independent, symmetric \( B \)-valued r.v.’s and let \( \{ \gamma_n, n \in \mathbb{N} \} \) be a sequence of positive real numbers. Let \( \vartheta \in \Phi_0 \) and \( \vartheta \sim A_2 \). If \( \left\| S_{kn} \right\| / \gamma_n, n \in \mathbb{N} \) is bounded in probability, then there exist constants \( a, b > 0 \) such that

\[
E \vartheta \left( \left\| S_{kn} \right\| \right) \leq a E \vartheta \left( \max_{1 \leq k \leq k_n} \left\| X_{nk} \right\| \right) + b \vartheta (\gamma_n) \quad \text{for all} \quad n \in \mathbb{N}.
\]

Proof. Let \( N_{kn} = \max_{1 \leq k \leq k_n} \left\| X_{nk} \right\| \). By \( \vartheta \in \Phi_0 \) and Lemma 4.2, we have for all \( x \geq 0 \) and \( n \in \mathbb{N} \)

\[
P(\vartheta(\left\| S_{kn} \right\| / 3) > \vartheta(x)) \leq P(\left\| S_{kn} \right\| / 3 > x) \leq P(N_{kn} > x) + 4P^2(\left\| S_{kn} \right\| > x)
\]

Hence

\[
P(\vartheta(\left\| S_{kn} \right\| / 3) > t) \leq P(\vartheta(N_{kn}) > t) + 4P^2(\vartheta(\left\| S_{kn} \right\| ) > t)
\]

for all \( t \in R_3 \) and \( n \in \mathbb{N} \). Now we prove (4.1) for \( t \in R_\vartheta \). First assume that \( t \in (\vartheta(0), \sup R_\vartheta) \cap R_\vartheta \). Then there exists \( a \geq 0 \), so that \( \lim_{x \to a^-} \vartheta(x) < t < \lim_{x \to a^+} \vartheta(x) \). (Define \( \lim_{x \to a^-} \vartheta(x) \) as \( \vartheta(0) \).) If \( \vartheta(a) > t \), then \( \bigcup_{m=1}^{\infty} \{ y : \vartheta(y) > \vartheta(a + 1/m) \} = \{ y : \vartheta(y) > t \} \) and \( \bigcup_{m=1}^{\infty} \{ y : \vartheta(y) \geq \vartheta(a + 1/m) \} = \{ y : \vartheta(y) \geq t \} \). On the other hand, if \( \vartheta(a) > t \), then \( \bigcup_{m=1}^{\infty} \{ y : \vartheta(y) > \vartheta(a - 1/m) \} = \{ y : \vartheta(y) > t \} \) and \( \bigcup_{m=1}^{\infty} \{ y : \vartheta(y) \geq \vartheta(a - 1/m) \} = \{ y : \vartheta(y) \geq t \} \). Hence, using continuity of probability and (4.1) for \( t \in R_\vartheta \), we have that (4.1) is true in this case as well. If \( 0 \leq t \leq \vartheta(0) \) or \( t \geq \sup R_\vartheta \), then (4.1) is obvious. Now, applying \( \vartheta \sim A_2 \), we get that there exists a constant \( c > 1 \) such that

\[
P(\vartheta(\left\| S_{kn} \right\| ) > ct) \leq P(\vartheta(N_{kn}) > t) + 4P^2(\vartheta(\left\| S_{kn} \right\| ) > t)
\]

for all \( t \geq 0 \). Integrating with respect to \( t \), we obtain

\[
\frac{1}{c} E \vartheta(\left\| S_{kn} \right\| ) \leq E \vartheta(N_{kn}) + 4 \int_0^\infty P^2(\vartheta(\left\| S_{kn} \right\| ) > t) dt.
\]

(4.3)

Since \( \left\| S_{kn} \right\| / \gamma_n, n \in \mathbb{N} \) is bounded in probability and \( \vartheta \sim A_2 \), therefore there exist constants \( A_1, A > 0 \) such that

\[
P(\left\| S_{kn} \right\| \geq A_1 \gamma_n) < \frac{1}{8c} \quad \text{and} \quad \vartheta(A_1 \gamma_n) \leq 4 \vartheta(\gamma_n)
\]

for all \( n \in \mathbb{N} \). Hence we have

\[
P(\vartheta(\left\| S_{kn} \right\| ) > A \vartheta(\gamma_n)) < \frac{1}{8c}.
\]
It follows that

\[ \int_0^\infty P^2(\theta(\|S_{kn}\|) > t) dt \leq \int_0^A \frac{1}{8c} P(\theta(\|S_{kn}\|) > t) dt \]

\[ \leq A \theta(\gamma_{kn}) + \frac{1}{8c} \mathbb{E} \theta(\|S_{kn}\|). \]  

(4.4)

Thus, by (4.3) and (4.4), we get Lemma 4.3. \(\square\)

The following lemma is a generalization of Lemma 2.1 of Gut (1992) and Lemma 2.7 (b) of Fazekas (1992).

**Lemma 4.4.** Let \(\{X_{nk}, n \in \mathbb{N}, k = 1, \ldots, k_n\}\) be an array of \(B\)-valued r.v.'s which is w.m.d. by the r.v. \(X\). If \(\theta \in \Phi_0\) then

\[ \frac{1}{k_n} \sum_{k=1}^{k_n} \mathbb{E} \theta(\|X_{nk}\|) \leq (1 + \gamma) \mathbb{E} \theta(|X|). \]  

(4.5)

**Proof.** Using \(\theta \in \Phi_0\) and (2.2), we have for all \(x \geq 0\)

\[ \frac{1}{k_n} \sum_{k=1}^{k_n} P(\theta(\|X_{nk}\|) > \theta(x)) \leq \frac{1}{k_n} \sum_{k=1}^{k_n} P(\|X_{nk}\| > x) \leq \gamma P(|X| > x) \leq \gamma P(\theta(|X|) > \theta(x)), \]

hence we obtain (4.5) for \(t \in R_\theta\). A standard calculation gives (4.5) for \(t \notin R_\theta\). \(\square\)

In the proof of Theorem 3.1 we shall apply Lemmas 4.3 and 4.4 for \(k_n = n\).

**Proof of Theorem 3.1.** First assume that \(X_{nk}\) are symmetric. Let \(\varepsilon > 0\). Using Lemma 4.2 and (2.2), we get

\[ P(\|S_n\| > \varepsilon \sqrt{\gamma} \theta(n)) \leq A \gamma n P(|X| > \varepsilon \mathbb{E} n) + B \gamma P(\|S_n\| > \varepsilon \theta(n)). \]  

(4.6)

To estimate the second term of (4.6) we can apply \(\theta \in \Phi_0\), \(\theta \sim \mathcal{A}_2\), Chebyshev’s inequality, Lemmas 4.3 and 4.4. Thus there exist \(\varepsilon', \gamma', a, b > 0\) such that for all \(n \geq n_0\)

\[ P\left(\frac{1}{\varepsilon} \|S_n\| > a \theta(n)\right) \leq P(\varepsilon' \theta(\|S_n\|) > \theta(\varepsilon(n))) \]

\[ \leq \varepsilon' \frac{\mathbb{E} \theta(\|S_n\|)}{\theta(\varepsilon(n))} \leq \frac{\varepsilon'}{\gamma(n)} (a \gamma' \mathbb{E} \theta(|X|) + b \theta(\gamma_{kn})). \]  

(4.7)

In formula (4.7) we can choose \(b\) such that

\[ b > \frac{a}{r} \gamma' \mathbb{E} \theta(|X|), \]  

(4.8)

where \(r\) is from (3.4). Now (4.6)–(4.8) imply that
\[ \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \mathbb{P}(\|S_n\| > \varepsilon x(n)) \leq A \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \mathbb{P}(|X| > \varepsilon x(n)) + \text{const.} \]
\[ + \text{const.} \sum_{n=n_0+1}^{\infty} \frac{\mu(n)}{n} \left( \frac{rm + \beta(\gamma_n)}{\beta(x(n))} \right)^{2^j}. \tag{4.9} \]

Since \( \varphi \sim A_2 \), there exists \( k > 0 \) such that \( E \varphi(|X|/\varepsilon) \leq kE \varphi(|X|) < \infty \). Thus, by Lemma 4.1 and (3.3), there exists \( n_1 \in \mathbb{N} \) such that
\[ \infty > \sum_{n=1}^{\infty} \beta(n-1) \mathbb{P} \left( \frac{|X|}{\varepsilon} > x(n) \right) \geq \text{const.} \sum_{n=n_1}^{\infty} \mu(n) \mathbb{P}(|X| > \varepsilon x(n)). \tag{4.10} \]

Then (4.9), (4.10) and (3.4) imply (3.5).

In the general case let \( X_{nk}^* \) be an independent copy of \( X_{nk} \) for any \( n \in \mathbb{N} \) and \( k = 1, \ldots, n \). Let \( X_{nk}^* = X_{nk} - X_{nk}' \), \( S_n' = \sum_{k=1}^{n} X_{nk}' \) and \( S_n^* = \sum_{k=1}^{n} X_{nk}^* = S_n - S_n' \).

Now prove that conditions of Theorem 3.1 hold for \( X_n^* \). Using (2.1) and (2.2), we get
\[ \frac{1}{n} \sum_{n=1}^{n} \mathbb{P} \left( \|X_n^*\| > t \right) \leq \frac{2}{n} \sum_{n=1}^{n} \mathbb{P} \left( \|X_{nk}\| > t \right) \leq 2 \mathbb{P}(2X > t) \]
for all \( t > 0 \), so \( \{X_{nk}^* : n \in \mathbb{N}, k = 1, \ldots, n\} \) is v.m.d. by \( 2X \). Moreover, it follows from \( \varphi, \theta \sim A_2 \) that \( E \varphi(2X) < \infty \) and \( E \theta(2X) < \infty \).

Since \( \|S_n'/\gamma_n, n \in \mathbb{N}\) \) is bounded in probability, using (2.1), for every \( h > 0 \) there exists \( q > 0 \) such that for all \( n \in \mathbb{N} \)
\[ 2h > 2 \mathbb{P}(\|S_n\| > q\gamma_n) \geq \mathbb{P}(\|S_n^*\| > 2q\gamma_n). \]

Thus \( \{\|S_n^*\|/\gamma_n, n \in \mathbb{N}\} \) is bounded in probability. Therefore the already known symmetric case implies
\[ \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \mathbb{P}(\|S_n^*\| > \varepsilon x(n)) < \infty \quad \text{for all} \quad \varepsilon > 0. \tag{4.11} \]

Now we turn to \( S_n \). \( \{\|S_n'/\gamma_n, n \in \mathbb{N}\} \) is bounded in probability as well, so there exists \( q' > 0 \) such that
\[ \mathbb{P}(\|S_n'/\gamma_n\| < q\gamma_n) > \frac{1}{2}. \tag{4.12} \]

Finally, (2.1), \( x(n)/\gamma_n \to \infty \) and (4.12) imply that for \( n \in \mathbb{N} \) large enough
\[ \mathbb{P}(\|S_n^*\| > \varepsilon x(n)) \geq \mathbb{P}(\|S_n'/\gamma_n\| < q\gamma_n) \mathbb{P}(\|S_n\| > 2\varepsilon x(n)) \]
\[ \geq \frac{1}{2} \mathbb{P}(\|S_n\| > 2\varepsilon x(n)) \geq \frac{1}{2} \mathbb{P}(\|S_n\| > 2\varepsilon x(n)). \]

This fact and (4.11) imply (3.5). \( \square \)

**Proof of Corollary 3.2.** In Theorem 3.1 put \( x(\gamma) = x^{t/2}, \varphi(x) = M^{t/2}(\psi(x^{t/r})) \), \( \theta(x) = x^t \) and \( \gamma_n = n^{1/s} \). Then
\[ \beta(k_n - 1) = E\varphi(x(k_n)) - \varphi(x(k_n - 1)) = M^{t/2}(n) - M^{t/2}(n - 1) - M^{t/2}(n) + M^{t/2}(n - 1) = 0 \]
\[ \text{for all } n \in \mathbb{N} \]. Using relation \( M^{t}(n) - M^{t}(n - 1) = \]
Furthermore, by (4.13)–(4.15), we have
\[ \frac{v}{M(n-1)} v^{r-1} dt, \]

it is easy to see that
\[ v(k_n) M^{v-1}(n-1) \leq M^v(n) - M^v(n-1) \leq v k_n^{2v-1} \quad \text{for all} \quad v \geq 1, \ n \in \mathbb{N}, \]  
(4.14)

and
\[ v(k_n) M^{v-1}(n) \leq M^v(n) - M^v(n-1) \leq v k_n \quad \text{for all} \quad 0 < v \leq 1, \ n \in \mathbb{N}. \]  
(4.15)

Let \( j \in \mathbb{N} \) be such that \( 2^j > r((r-1) \vee 1)/(rs - r) \). Then, using (4.13)–(4.15), we have
\[ \sum_{k=1}^{\infty} \frac{\beta(k_n - 1)}{k_n} \left( \frac{r k_n + \gamma(k_n)}{\varphi(k_n)} \right)^{2^j} \leq \begin{cases} \text{constant} \sum_{n=1}^{\infty} n^{r-2-(sr/r-1)2^j} < \infty, & \text{if } r > 2, \\ \text{constant} \sum_{n=1}^{\infty} n^{-(sr/r-1)2^j} < \infty, & \text{if } 0 < r \leq 2. \end{cases} \]

It is easy to see that the other conditions of Theorem 3.1 are satisfied as well. Thus
\[ \sum_{n=1}^{\infty} \frac{\beta(k_n - 1)}{k_n} P\left( \| S_{k_n} \| > \varepsilon k_n^{r/2} \right) < \infty \quad \text{for all} \quad \varepsilon > 0. \]

Furthermore, by (4.13)–(4.15), we have \( \beta(k_n - 1)/k_n \geq \text{constant} \ (M(n))^{r/2-1} \) which implies the statement. \( \square \)

**Proof of Corollary 3.3.** In Theorem 3.1 put \( \varphi(x) = x^{1/r}, \ \varphi(x) = M_r(\varphi(x')) \), \( \varphi(x) = x^r \) and \( \gamma_n = n^{1/s} \). It is easy to see that the conditions of Theorem 3.1 are satisfied. Thus Theorem 3.1 implies the statement, because in this case \( \beta(k_n - 1)/k_n = k_n^{r-2} \) and \( \beta(m-1) = 0 \) if \( k_n < m < k_{n+1} \). \( \square \)

5. Special cases of the main theorem

\( B \) is said to be of \((\text{Rademacher}) \) type \( p \) \((0 < p \leq 2)\) if there exists a \( c > 0 \) such that
\[ E \left\| \sum_{i=1}^{n} X_i \right\|^p \leq c \sum_{i=1}^{n} E \left\| X_i \right\|^p \]  
(5.1)

for every independent \( B \)-valued r.v.’s \( X_1, \ldots, X_n \) with \( E \left\| X_i \right\|^p < \infty \) (and \( E X_i = 0 \) if \( p \geq 1), \ i = 1, \ldots, n. \)

The following remark shows that in Theorem 3.1 we can write moment conditions instead of the boundedness of \( \left\| S_{k_n}/\gamma_{k_n} \right\|, n \in \mathbb{N} \) if \( B \) is of type \( p. \)

**Remark 5.1.** Let \( B \) be of type \( p \) for some \( 0 < p \leq 2 \). Let \( \{X_{nk}, n \in \mathbb{N}, k = 1, \ldots, k_n\} \) be an array of rowwise independent \( B \)-valued r.v.’s which is w.m.d. by the r.v. \( X \). Assume that \( EX_{nk} = 0 \) \((k = 1, \ldots, k_n)\) when \( p \geq 1 \). If \( E|X|^p < \infty \) then \( \left\| S_{k_n}/k_n^{1/p}, n \in \mathbb{N} \right\| \) is bounded in probability.

**Proof.** Using (5.1) and Lemma 4.4, we have
\[ E \left\| S_{k_n} \right\|^p \leq c \sum_{k=1}^{k_n} E \left\| X_{nk} \right\|^p \leq c(1 \vee \gamma)k_n E|X|^p. \]

So \( \left\| S_{k_n}/k_n, n \in \mathbb{N} \right\| \) is bounded in probability. \( \square \)
The following corollary is a version of Corollary 4.2 of Hu et al. (1999).

**Corollary 5.2.** Let $B$ be of type $p$ for some $0 < p \leq 2$. Let $\{X_{nk}, n \in \mathbb{N}, k = 1, \ldots, k_n\}$ be an array of rowwise independent $B$-valued r.v.’s which is w.m.d. by the r.v. $X$. Let $r \in \mathbb{R}$, $0 < t < p$ and $M_r \circ \psi \sim \Delta_2$. If $EX_{nk} = 0$ for all $n \in \mathbb{N}, k = 1, \ldots, k_n$, $EM_r(\psi(|X|^t)) < \infty$ and $E|X|^p < \infty$, then

$$
\sum_{n=1}^{\infty} k_n^{t-1} P(\|S_{kn}\| > \varepsilon k_n^{1/p}) < \infty \quad \text{for all} \quad \varepsilon > 0.
$$

**Proof.** It follows from Remark 5.1 that $\{\|S_{kn}\|/k_n^{1/p}, n \in \mathbb{N}\}$ is bounded in probability. Hence conditions of Theorem 3.3 are satisfied. □

The following three theorems are due to Fazekas (1992). We shall prove that they are special cases of Theorem 3.1.

**Theorem 5.3 (Fazekas, 1992, Theorem 3.1).** Let $0 < p \leq 2$, $s > p$, $rp > s$ and let $B$ be of type $p$. Let $\{X_{nk}, n \in \mathbb{N}, k = 1, \ldots, n\}$ be an array of rowwise independent $B$-valued r.v.’s which is w.m.d. by the r.v. $X$. Assume that $EX_{nk} = 0$ $(k = 1, \ldots, n)$ when $p \geq 1$. If $E|X|^t < \infty$, then

$$
\sum_{n=1}^{\infty} n^{r-2} P(\|S_n\| > \varepsilon n^{r/s}) < \infty \quad \text{for all} \quad \varepsilon > 0.
$$

**Proof.** In Theorem 3.1 put $\alpha(x) = x^{r/s}$, $\varphi(x) = \vartheta(x) = x^s$ and $\gamma_n = n^{1/p}$. Let $j \in \mathbb{N}$ such that $2^j > rp/(rp - s)$. By Remark 5.1, $\{\|S_n\|/n^{1/p}, n \in \mathbb{N}\}$ is bounded in probability. It is easy to see that the other conditions of Theorem 3.1 hold true as well. □

**Theorem 5.4 (Fazekas 1992, Theorem 3.5 and Jain 1975, Theorem 3.3).** Let $\{X_{nk}, n \in \mathbb{N}, k = 1, \ldots, n\}$ be an array of rowwise independent $B$-valued r.v.’s which is w.m.d. by the r.v. $X$. Let $\alpha, \varphi \in \Phi_0$, which are strictly increasing, $R_2 = R_\varphi = [0, \infty)$ and $\varphi \sim \Delta_2$. Let $\beta(n) = \varphi(\alpha(n + 1)) - \varphi(\alpha(n))$ such that for some $c_1, c_2 > 0$

$$
c_1 \leq c_2 \beta(n + 1) \leq \beta(n) \quad \text{for all} \quad n \in \mathbb{N}.
$$

Let $E \varphi(|X|) < \infty$. Assume that there exists a sequence $\{\gamma_n, n \in \mathbb{N}\}$ of positive real numbers such that $\{\|S_n\|/\gamma_n, n \in \mathbb{N}\}$ is bounded in probability, moreover there exists $\delta > 0$ such that

$$
\frac{n \vee \varphi(\gamma_n)}{\varphi(\alpha(n))} = O((\log n)^{-\delta} \wedge (\beta(n))^{-\delta}).
$$

Then

$$
\sum_{n=1}^{\infty} \frac{\beta(n)}{n} P(\|S_n\| > \varepsilon \alpha(n)) < \infty \quad \text{for all} \quad \varepsilon > 0.
$$

**Proof.** In Theorem 3.1 put $\vartheta = \varphi$ and choose $j \in \mathbb{N}$ such that $2^j > 2/\delta$. Then, using (5.2) and $1/\beta(n) \leq 1/c_1$, we get for some $m_0 \in \mathbb{N}$ that
Acknowledgements.
The author would like to thank István Fazekas for several helpful discussions and for his attention to my paper.

\[
\sum_{n=1}^{\infty} \frac{\beta(n)}{n} \left( \frac{r n + 9 \gamma_n}{3}(n) \right)^{2^j} \leq \text{const.} + \text{const.} \sum_{n=n_0}^{\infty} \frac{\beta(n)}{n} \left( \frac{r + 1}{(\beta(n) \log n)^{q/2}} \right) \leq \text{const.} + \text{const.} \sum_{n=n_0}^{\infty} n^{-1}(\log n)^{-2^{2^j}-1} < \infty.
\]

It follows from (5.2) that const. \((\log n)^q \leq \phi(\gamma(n))/\phi(\gamma_n)\) for \(n \in \mathbb{N}\) large enough, hence \(
\phi(\gamma(n))/\phi(\gamma_n) \rightarrow \infty\). This fact and \(\varphi \sim A_2\) imply that \(\gamma(n)/\gamma_n \rightarrow \infty\). Consequently, Theorem 3.1 implies (5.3). \(\square\)

**Theorem 5.5** (Fazekas 1992, Theorem 6.2). Let \(\{X_{nk}, n \in \mathbb{N}, k = 1, \ldots, k_n\}\) be an array of rowwise independent B-valued r.v.’s which is w.m.d. by the r.v. \(X\). Let \(0 < p \leq 2, r \geq 1, t > 0\) and \(s \geq p\). Suppose that \(r > t/p\) if \(s > 1\) while \(r > t/s\) if \(s \leq 1\). Assume that
\[
\limsup_{n \to \infty} \frac{k_n}{M(n-1)} < \infty \quad \text{if} \quad r > 2.
\]
Let \(M \circ \psi \sim A_2\) and \(B\) be of type \(p\). Assume that \(EX_{nk} = 0\) \((k = 1, \ldots, k_n)\) in case \(p \geq 1\). If
\[
EM^{r/2}(\psi(|X|^q)) < \infty \quad \text{and} \quad E|X|^r < \infty,
\]
then
\[
\sum_{n=1}^{\infty} (M(n))^{q/2-1} P\left( \left\| S_{k_n} \right\| > c k_n^{q/4} \right) < \infty \quad \text{for all} \quad c > 0.
\]

**Proof.** Let \(q = p\) if \(s > 1\) while \(q = s\) if \(s \leq 1\). Then \(rq > t\), \(B\) is of type \(q\) and \(E|X|^q < \infty\). Hence, using Remark 5.1, we get that \(\{\left\| S_{k_n} \right\|/k_n^{q/4}, n \in \mathbb{N}\}\) is bounded in probability. On the other hand \(\limsup_{n \to \infty} k_n/M(n-1) < \infty\) implies the boundedness of \(\{M(n)/M(n-1), n \in \mathbb{N}\}\). So all conditions of Corollary 3.2 are satisfied. \(\square\)

**Remark 5.6.** Theorem 5.5 can be applied e.g., for \(k_n = d^n\) and \(k_n = n^d\), where \(d\) is a fixed positive integer.

**Acknowledgements.** The author would like to thank István Fazekas for several helpful discussions and for his attention to my paper.

**References**


