ANNALES UNIV. SCI. BUDAPEST., **43** (2000), 61–78

A GENERAL APPROACH TO STRONG LAWS OF LARGE NUMBERS FOR FIELDS OF RANDOM VARIABLES

By

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(Received July 7, 1999)

1. Introduction and Notation

There are several methods to obtain almost sure (a.s.) convergence results for random fields (see e.g. [14], [9], [8], [5] and the literature cited there). The aim of our paper is to present a general approach to obtain strong laws of large numbers (SLLN) for random fields. Our method is an extension of the one given in [6]. In [6] only random sequences (i.e. not fields) were considered.

The paper is organized as follows. Section 2 contains the main result (Theorem 3). Once a maximal inequality is known, Theorem 3 easily implies an SLLN and it helps to obtain appropriate normalizing constants in the SLLN. The remaining sections contain applications. In Section 3 an SLLN is presented for logarithmically weighted sums. We remark that such kind of SLLN's are useful to prove almost sure central limit theorems (see e.g. [3]). In Section 4 the case of fields with superadditive moment structure is studied. In Section 5 a Brunk–Prokhorov type SLLN is presented. Section 6 is devoted to mixingales.

In the following $\mathbb{N}_0$ and $\mathbb{N}$ denote the set of nonnegative and positive integers, respectively. Let d be a fixed positive integer. Throughout the paper I, J, K, L, M and N denote elements of $\mathbb{N}_0^d$ (in particular, elements of $\mathbb{N}^d$). If an element of $\mathbb{N}_0^d$ (or $\mathbb{N}^d$) is denoted by a capital letter, then its coordinates are denoted by the lower case of the same letter, i.e. N always means the vector

Supported by the Hungarian Ministry of Culture and Education under Grant No. FKFP 0121/1999.

$(n_1, \dots, n_d) \in \mathbb{N}_0^d$. We also use $\mathbf{1} = (1, \dots, 1) \in \mathbb{N}^d$ and $\mathbf{0} = (0, \dots, 0) \in \mathbb{N}_0^d$. In $\mathbb{N}_0^d$ we consider the coordinate-wise partial ordering: $M \leq N$ means $m_i \leq n_i$, $i = 1, \dots, d$ ($M < N$ means $M \leq N$ and $N \neq M$). $N \to \infty$ is interpreted as $n_i \to \infty$, $i = 1, \dots, d$, $\lim_N a_N$ is meant in this sense. In $\mathbb{N}_0^d$ the maximum is defined coordinate-wise (actually we shall use it only for rectangles). If $N = (n_1, \dots, n_d) \in \mathbb{N}_0^d$ then $\langle N \rangle = \prod_{i=1}^d n_i$.

A numerical sequence a_N, $N \in \mathbb{N}_0^d$, is called d-sequence. If a_N is a d-sequence then its difference sequence, i.e. the d-sequence b_N for which $\sum_{M \leq N} b_M = a_N$, $N \in \mathbb{N}^d$, will be denoted by Δa_N.

We shall say that a d-sequence a_N is of product type if $a_N = \prod_{i=1}^d a_{n_i}^{(i)}$, where $a_{n_i}^{(i)}$ ($n_i = 0, 1, 2, \dots$) is a (single) sequence for each $i = 1, \dots, d$. Our consideration will be confined to normalizing constants of product type: b_N will always denote $b_N = \prod_{i=1}^d b_{n_i}^{(i)}$, where $b_{n_i}^{(i)}$, $n_i = 0, 1, 2, \dots$, is a nondecreasing sequence of positive numbers for each $i = 1, \dots, d$. In this case we shall say that b_N is a positive nondecreasing d-sequence of product type. Moreover, if for each $i = 1, \dots, d$ the sequence $b_{n_i}^{(i)}$ is unbounded, then b_N is called positive, nondecreasing, unbounded d-sequence of product type.

The random field will be denoted by X_N, $N \in \mathbb{N}_0^d$. S_N is the partial sum: $S_N = \sum_{M \leq N} X_M$ for $N \in \mathbb{N}_0^d$. As X_N is a field with lattice indices we shall say that X_N, $N \in \mathbb{N}_0^d$, is a d-sequence of random variables (r.v.'s). Remark that a sum or a maximum over the empty set will be interpreted as zero (i.e. $\sum_{N \in \mathcal{H}} X_N = \max_{N \in \mathcal{H}} X_N = 0$ if $\mathcal{H} = \emptyset$). As usual, $\log^+(x) = \max\{1, \log(x)\}$, $x > 0$.

2. The Basic SLLN

The proposition and lemma below are useful for proving Theorem 3. Proposition 1 and its proof are straightforward generalizations of Theorem 1.1 and its proof in [6]. Note that there are several other ways to obtain maximal inequalities of this type: see for example [8].

PROPOSITION 1. (Hájek–Rényi type maximal inequality.) *Let $N \in \mathbb{N}^d$ be fixed. Let r be a positive real number, a_N be a nonnegative d-sequence.*

Suppose that b_M is a positive, nondecreasing d-sequence of product type. Then

$$\mathbb{E}\left\{\max_{L\leq M}|S_L|^r\right\}\leq\sum_{L\leq M}a_L\qquad\forall M\leq N$$

implies

$$\mathbb{E}\left\{\max_{M\leq N}\left|\frac{S_M}{b_M}\right|^r\right\}\leq 4^d\sum_{M\leq N}\frac{a_M}{b^r_M}.$$

PROOF. Without loss of generality we can assume that $b_{\mathbf{1}}=1$. Fix an $N\in\mathbb{N}^d$ and for a moment a real number $c>1$. For $I=(i_1,\dots,i_d)\in\mathbb{N}_0^d$ let us define the set

$$\mathcal{A}_I=\{J\in\mathbb{N}^d\ :\ J\leq N \text{ and } c^{i_k}\leq b^{(k)}_{j_k}<c^{i_k+1},\quad k=1,\dots,d\}.$$

Now we can form

$$D_I=\sum_{J\in\mathcal{A}_I}a_J\quad\text{and}\quad K=\max\{I\ :\ \mathcal{A}_I\neq\emptyset\},$$

where D_I as we mentioned above is considered to be zero if $\mathcal{A}_I=\emptyset$. Note that K is well defined because of product form of b_N. It is easy to see that each nonempty $\mathcal{A}_I$ has a maximal element. Therefore if $\mathcal{A}_I\neq\emptyset$ let

$$M_I=\max\{J\ :\ J\in\mathcal{A}_I\}$$

otherwise set $M_I=\mathbf{0}$. Since $\bigcup_{I\leq K}\mathcal{A}_I$ covers the rectangle $\{M\in\mathbb{N}^d\ :\ M\leq\leq N\}$ so

$$\mathbb{E}\left\{\max_{M\leq N}\left|\frac{S_M}{b_M}\right|^r\right\}\leq\sum_{J\leq K}\mathbb{E}\left\{\max_{I\in A_J}\left|\frac{S_I}{b_I}\right|^r\right\}.$$

By the definition of $\mathcal{A}_I$, M_I and D_I we get

$$\sum_{J\leq K}\mathbb{E}\left\{\max_{I\in A_J}\left|\frac{S_I}{b_I}\right|^r\right\}\leq\sum_{J\leq K}\left\{\prod_{m=1}^d c^{-rj_m}\right\}\mathbb{E}\left\{\max_{I\in\mathcal{A}_J}|S_I|^r\right\}\leq$$

$$\leq\sum_{J\leq K}\left\{\prod_{m=1}^d c^{-rj_m}\right\}\mathbb{E}\left\{\max_{I\leq M_J}|S_I|^r\right\}\leq\sum_{J\leq K}\left\{\prod_{m=1}^d c^{-rj_m}\right\}\sum_{I\leq M_J}a_I\leq$$

$$\leq\sum_{J\leq K}\left\{\prod_{m=1}^d c^{-rj_m}\right\}\sum_{I\leq J}D_I\leq\sum_{I\leq K}D_I\sum_{I\leq J\leq K}\left\{\prod_{m=1}^d c^{-rj_m}\right\}\leq$$

$$\le \sum_{I\le K} D_I \prod_{m=1}^{d}\left\{\sum_{j=i_m}^{k_m} c^{-rj}\right\} \le \sum_{I\le K} D_I \prod_{m=1}^{d} \frac{c^{-ri_m}}{1-c^{-r}} \le$$

$$\le \left\{\frac{c^r}{1-c^{-r}}\right\}^d \sum_{I\le K} D_I \prod_{m=1}^{d} c^{-r(i_m+1)} \le$$

$$\le \left\{\frac{c^r}{1-c^{-r}}\right\}^d \sum_{I\le K}\left\{\sum_{J\in\mathcal{A}_I} a_J\right\} \prod_{m=1}^{d} c^{-r(i_m+1)} \le$$

$$\le \left\{\frac{c^r}{1-c^{-r}}\right\}^d \sum_{I\le K}\sum_{J\in\mathcal{A}_I} \frac{a_J}{b_J^r} \le \left\{\frac{c^r}{1-c^{-r}}\right\}^d \sum_{J\le N} \frac{a_J}{b_J^r}.$$

This proves the proposition because $\inf_{c>1} \frac{c^r}{1-c^{-r}} = 4$. ∎

LEMMA 2. *Let a_N be a nonnegative d-sequence and let b_N be a positive, nondecreasing, unbounded d-sequence of product type. Suppose that $\sum_N \frac{a_N}{b_N^r} < +\infty$ with a fixed real $r > 0$. Then there exists a positive, nondecreasing, unbounded d-sequence β_N of product type for which*

$$\lim_N \frac{\beta_N}{b_N} = 0 \quad \textit{and} \quad \sum_N \frac{a_N}{\beta_N^r} < +\infty.$$

PROOF. Clearly it is enough to prove for $r = 1$. In case of $d = 1$ one can find our proposition in [6, Lemma 2.2]. Let $d \ge 2$. Then

$$+\infty > \sum_N \frac{a_N}{b_N} = \sum_{n_1} \frac{1}{b_{n_1}^{(1)}} \sum_{n_2,\dots,n_d} \frac{a_N}{\prod_{m=2}^d b_{n_m}^{(m)}} = \sum_{n_1} \frac{1}{b_{n_1}^{(1)}} T_{n_1}$$

with $T_{n_1} = \sum_{n_2,\dots,n_d} \frac{a_N}{\prod_{m=2}^d b_{n_m}^{(m)}}$. Applying the above mentioned lemma of [6], we get that there exists an unbounded, positive, nondecreasing sequence $\beta_n^{(1)}$ so that

$$\lim_n \frac{\beta_n^{(1)}}{b_n^{(1)}} = 0 \quad \text{and} \quad \sum_{n_1} \frac{1}{\beta_{n_1}^{(1)}} T_{n_1} < +\infty.$$

If we have already obtained $\beta_n^{(m)}$ for $m = 1,\dots,k$, $k < d$ then replacing in the above procedure b_N by $\prod_{m=1}^k \beta_{n_m}^{(m)} \prod_{m=k+1}^d b_{n_m}^{(m)}$ and coordinate 1 by coordinate $k+1$ we get an appropriate $\beta_n^{(k+1)}$. Finally, by setting $\beta_N = \prod_{m=1}^{d} \beta_{n_m}^{(m)}$, it obviously satisfies the requirements. ∎

The following theorem is an extension of Theorem 2.1 of [6].

THEOREM 3. *Let a_N, b_N be nonnegative d-sequences and let $r > 0$. Suppose that b_N is a positive, nondecreasing, unbounded d-sequence of product type. Then*

$$\sum_N \frac{a_N}{b_N^r} < +\infty \quad \textit{and} \quad \mathbb{E}\left\{\max_{M \le N} |S_M|^r\right\} \le \sum_{M \le N} a_M \quad \forall N \in \mathbb{N}^d$$

imply

$$\lim_N \frac{S_N}{b_N} = 0 \quad \textit{a.s.}$$

PROOF. Let β_N be the sequence obtained in the previous lemma. According to Proposition 1:

$$\mathbb{E}\left\{\max_{M \le N} \left|\frac{S_M}{\beta_M}\right|^r\right\} \le 4^d \sum_{M \le N} \frac{a_M}{\beta_M^r} \quad \forall N \in \mathbb{N}^d.$$

Hence

$$\mathbb{E}\left\{\sup_{n_d} \ldots \sup_{n_1} \left|\frac{S_N}{\beta_N}\right|^r\right\} \le 4^d \sum_N \frac{a_N}{\beta_N^r}.$$

Since

$$\sup_{n_d} \ldots \sup_{n_1} \left|\frac{S_N}{\beta_N}\right|^r = \sup_N \left|\frac{S_N}{\beta_N}\right|^r$$

it follows from the foregoing that

$$\sup_N \left|\frac{S_N}{\beta_N}\right|^r < +\infty \qquad \text{a.s.}$$

We have

$$\left|\frac{S_N}{b_N}\right| = \frac{\beta_N}{b_N} \left|\frac{S_N}{\beta_N}\right| \le \frac{\beta_N}{b_N} \sup_K \left|\frac{S_K}{\beta_K}\right|.$$

This proves the theorem because $\lim_N \frac{\beta_N}{b_N} = 0$. ∎

DEFINITION 4. A function g on $\mathbb{N}^d \times \mathbb{N}^d$ is said to be *superadditive* if $g\left(I, (j_1, \ldots, j_{m-1}, k, j_{m+1}, \ldots, j_d)\right) + g\left(i_1, \ldots, i_{m-1}, k+1, i_{m+1}, \ldots, i_d), J\right)$

can be majorized by $g(I,J)$ for any $m = 1,\dots,d$ and for any $i_m \le k < j_m$. A d-sequence of random variables is said to have r-th moment function of superadditive structure ($MFSS$) if

$$\mathbb{E}\left\{\left|\sum_{I\le K\le J} X_K\right|^r\right\} \le g(I,J)^\alpha \qquad \forall I,J\in\mathbb{N}^d,$$

where g is superadditive on $\mathbb{N}^d\times\mathbb{N}^d$, $r > 0$ and $\alpha > 1$. Remark that the notion of r-th $MFSS$ was used by Móricz in [11].

REMARK 5. Maximal inequalities play important role in proving SLLN's. We shall frequently use the following result of Móricz (see [9, Corollary 1] or [12, Theorem 7]):

Suppose that $r \ge 1$ and X_N has r-th $MFSS$. Then there is a constant $A_{r,\alpha,d}$ for which

$$\mathbb{E}\left\{\max_{K\le N} |S_K|^r\right\} \le A_{r,\alpha,d}\, g(1,N)^\alpha \quad \forall N\in\mathbb{N}^d.$$

The reader can verify that the above proposition is true in the case of $0 < r < 1$, too. ∎

3. Logarithmically Weighted Sums

Móri in [15, Theorem 1] proved that the sequence

$$\frac{1}{\log^+ n}\sum_{k=1}^{n}\frac{X_k}{k}$$

converges a.s. to zero under general assumptions. With their general method in [6] Fazekas and Klesov proved a special case of Móri's theorem. Now we extend this case to fields of random variables. Our method is a generalization that of [6]. Our Lemma 6 and Theorem 7 are extensions of Lemma 9.1 and Theorem 9.1 of [6], respectively. In the lemma below $[x]$, $x \ge 0$ denotes the integer part of x, i.e. $[x]$ is the largest integer for which $[x] \le x$.

LEMMA 6. *(a) Let* $n \in \mathbb{N}$ *and* $0 < \beta < 1$. *Then there is a constant* $C_{d,\beta}$ *depending only on* d *and* β *such that:*

$$\sum_{m_1=1}^{n} \sum_{m_2=1}^{\left[\frac{n}{m_1}\right]} \cdots \sum_{m_d=1}^{\left[\frac{n}{m_1 m_2 \cdots m_{d-1}}\right]} \frac{1}{\langle M\rangle^{1-\beta}} \leq C_{d,\beta}\, n^{\beta} (\log^+ n)^{d-1}.$$

(b) Let $0 < \beta < 1$, $1 < \gamma < 2$, $I, M, J \in \mathbb{N}^d$, $I \leq M \leq J$. *Then there is a constant* $C_{d,\beta}$ *depending only on* d *and* β *such that:*

$$\sum_{I \leq M \leq J} \sum_{\substack{I \leq K \leq J \\ \langle K\rangle \leq \langle M\rangle}} \frac{1}{\langle M\rangle^{1+\beta}} \frac{1}{(\log^+\langle M\rangle)^{d-1}} \frac{1}{\langle K\rangle^{1-\beta}} \leq C_{d,\beta} \left\{ \sum_{I \leq M \leq J} \frac{1}{\langle M\rangle} \right\}^{\gamma}.$$

PROOF. (a) The case $d = 1$ is well known from elementary analysis. We prove by induction on d. Suppose that the statement is true for $d = f$. Let $n \in \mathbb{N}$ and $0 < \beta < 1$. Then

$$\sum_{m_1=1}^{n} \sum_{m_2=1}^{\left[\frac{n}{m_1}\right]} \cdots \sum_{m_{f+1}=1}^{\left[\frac{n}{m_1 m_2 \cdots m_f}\right]} \frac{1}{\langle M\rangle^{1-\beta}} =$$

$$= \sum_{m_1=1}^{n} \frac{1}{m_1^{1-\beta}} \sum_{m_2=1}^{\left[\frac{n}{m_1}\right]} \cdots \sum_{m_{f+1}=1}^{\left[\frac{n}{m_1 m_2 \cdots m_f}\right]} \frac{1}{(m_2 \cdots m_f)^{1-\beta}}.$$

Now applying the hypothesis for $\left[\frac{n}{m_1}\right]$ we get that the above expression is majorized by:

$$C_{f,\beta} \sum_{m_1=1}^{n} \frac{1}{m_1^{1-\beta}} \left[\frac{n}{m_1}\right]^{\beta} \left\{\log^+ \left[\frac{n}{m_1}\right]\right\}^{f-1} \leq C_{f,\beta} n^{\beta} (\log^+ n)^{f-1} \sum_{m_1=1}^{n} \frac{1}{m_1} \leq$$

$$\leq C_{f,\beta} n^{\beta} (\log^+ n)^{f-1} C \log^+ n$$

with certain $C > 0$ (here we used the fact $\left[\frac{1}{c}\left[\frac{a}{b}\right]\right] = \left[\frac{a}{bc}\right]$ for $a, b, c \in \mathbb{N}$).

(b) In case $\sum_{I\le M\le J}\frac{1}{\langle M\rangle}\le 1$ we get that

$$\sum_{I\le M\le J}\sum_{\substack{I\le K\le J\\ \langle K\rangle\le\langle M\rangle}}\frac{1}{\langle M\rangle^{1+\beta}(\log^+\langle M\rangle)^{d-1}\langle K\rangle^{1-\beta}}\le$$

$$\le\sum_{I\le M\le J}\sum_{\substack{I\le K\le J\\ \langle K\rangle\le\langle M\rangle}}\frac{1}{\langle M\rangle^{1+\beta}\langle K\rangle^{1-\beta}}\le$$

$$\le\sum_{I\le M\le J}\sum_{\substack{I\le K\le J\\ \langle K\rangle\le\langle M\rangle}}\frac{1}{\langle M\rangle\langle K\rangle}\le\left\{\sum_{I\le M\le J}\frac{1}{\langle M\rangle}\right\}^2\le\left\{\sum_{I\le M\le J}\frac{1}{\langle M\rangle}\right\}^\gamma.$$

In case $\sum_{I\le M\le J}\frac{1}{\langle M\rangle}>1$ using part (a) and the simple fact, that

$$\sum_{m_1=1}^{n}\sum_{m_2=1}^{\left[\frac{n}{m_1}\right]}\cdots\sum_{m_d=1}^{\left[\frac{n}{m_1m_2\cdots m_{d-1}}\right]}\frac{1}{\langle M\rangle^{1-\beta}}=\sum_{\substack{M\in\mathbb{N}^d\\ \langle M\rangle\le n}}\frac{1}{\langle M\rangle^{1-\beta}}$$

holds for all $n,d\in\mathbb{N}$, we get that

$$\sum_{I\le M\le J}\frac{1}{\langle M\rangle^{1+\beta}(\log^+\langle M\rangle)^{d-1}}\sum_{\substack{I\le K\le J\\ \langle K\rangle\le\langle M\rangle}}\frac{1}{\langle K\rangle^{1-\beta}}\le$$

$$\le C_{d,\beta}\sum_{I\le M\le J}\frac{1}{\langle M\rangle^{1+\beta}(\log^+\langle M\rangle)^{d-1}}\langle M\rangle^{\beta}(\log^+\langle M\rangle)^{d-1}=$$

$$=C_{d,\beta}\sum_{I\le M\le J}\frac{1}{\langle M\rangle}\le C_{d,\beta}\left\{\sum_{I\le M\le J}\frac{1}{\langle M\rangle}\right\}^\gamma.$$ ∎

THEOREM 7. *Let $X_N, N\in\mathbb{N}^d$, be a d-sequence of random variables and suppose that for some $C>0,\beta>0$*

$$|\mathbb{E}(X_KX_L)|\le C\left\{\frac{\langle K\rangle}{\langle L\rangle}\right\}^\beta\frac{1}{(\log^+\langle L\rangle)^{d-1}}\quad \text{if } \langle K\rangle\le\langle L\rangle,$$

Then

$$\lim_N\frac{1}{\prod_{i=1}^d\log^+n_i}\sum_{K\le N}\frac{X_K}{\langle K\rangle}=0 \text{ a.s.}$$

PROOF. Clearly it is enough to prove for $0 < \beta < 1$. Let $I, J \in \mathbb{N}^d, I \leq J$. Using the assumptions we get:

$$\mathbb{E}\left\{\left|\sum_{I\leq K\leq J}\frac{X_K}{\langle K\rangle}\right|^2\right\} \leq 2\sum_{I\leq L\leq J}\sum_{\substack{I\leq K\leq J\\ \langle K\rangle\leq\langle L\rangle}}\frac{1}{\langle K\rangle\langle L\rangle}|\mathbb{E}(X_K X_L)| \leq$$

$$\leq 2C\sum_{I\leq L\leq J}\sum_{\substack{I\leq K\leq J\\ \langle K\rangle\leq\langle L\rangle}}\frac{1}{\langle K\rangle^{1-\beta}\langle L\rangle^{1+\beta}(\log^+\langle L\rangle)^{d-1}}.$$

Let $1 < \gamma < 2$. It follows from Lemma 6(b) that

$$\mathbb{E}\left\{\left|\sum_{I\leq K\leq J}\frac{X_K}{\langle K\rangle}\right|^2\right\} \leq D_{d,\beta}\left\{\sum_{I\leq L\leq J}\frac{1}{\langle L\rangle}\right\}^{\gamma},$$

where $D_{d,\beta} > 0$ depends only on d and β. Now, from Remark 5 we get that

$$\mathbb{E}\left\{\max_{I\leq J}\left|\sum_{K\leq I}\frac{X_K}{\langle K\rangle}\right|^2\right\} \leq C_{d,\beta,\gamma}\left\{\sum_{K\leq J}\frac{1}{\langle K\rangle}\right\}^{\gamma} \qquad \forall J,$$

where $C_{d,\beta,\gamma} > 0$ depends only on d, β and γ. From the Hölder inequality we have:

$$\mathbb{E}\left\{\max_{I\leq J}\left|\sum_{K\leq I}\frac{X_K}{\langle K\rangle}\right|^{\frac{2}{\gamma}}\right\} \leq (C_{d,\beta,\gamma})^{\frac{1}{\gamma}}\sum_{K\leq J}\frac{1}{\langle K\rangle} \qquad \forall J.$$

Now we can apply Theorem 3 because

$$\sum_N \frac{1}{(\prod_{m=1}^d \log n_m)^{\frac{2}{\gamma}}}\frac{1}{\langle N\rangle} < +\infty. \qquad \blacksquare$$

Now we state some analogues of Theorem 7.

REMARK 8. Let X_N be an orthogonal d-sequence of random variables, $r > 0$ and $s > \frac{1+r}{2}$. Suppose that for some $C > 0$

$$\mathbb{E}(X_K^2) \leq C\langle K\rangle^r.$$

Then for any $\rho > 1$

$$\lim_N \frac{1}{\left\{\prod_{i=1}^d \log^+ n_i\right\}^\rho} \sum_{K \le N} \frac{X_K}{\langle K\rangle^s} = 0 \quad \text{a.s.}$$

For the proof one can use the d-multiple version of the Rademacher–Menšov inequality [9, Corollary 3a]. ∎

REMARK 9. Let $0 < r < 1$ and $0 < s \le \frac{2}{3-r}$. Suppose that for some $C > 0$

$$|\mathbb{E}(X_K X_L)| \le \frac{C\langle K\rangle^{sr}}{\langle L\rangle^s (\log^+\langle L\rangle)^{d-1}} \quad \text{if } \langle K\rangle \le \langle L\rangle.$$

Then

$$\lim_N \frac{1}{\langle N\rangle^{1-s}} \sum_{K \le N} \frac{X_K}{\langle K\rangle^s} = 0 \text{ a.s.}$$

The proof is similar to that of Theorem 7. ∎

4. Sequences with superadditive moment structure

In this section we prove a Marcinkiewicz–Zygmund type SLLN for d-sequences with superadditive moment structure. Our Proposition 11 is a generalization of Theorem 8.1 of [6]. For the sake of completeness we start with a simple technical lemma on partial summation.

LEMMA 10. *Let a_N, b_N be nonnegative d-sequences such that $b_N = \frac{1}{\langle N\rangle^\alpha}$ for some $\alpha > 0$. Then*

$$\sum_N (-1)^d \Lambda_N \Delta b_{N+\mathbf{1}} < +\infty$$

implies

$$\sum_N a_N b_N < +\infty,$$

where $\Lambda_N = \sum_{M \le N} a_M$. ∎

PROPOSITION 11. *Let $r > 0, \alpha > 1$ and suppose that X_N has r-th $MFSS$ and $\Delta g(\mathbf{1}, N)^\alpha$ is nonnegative for any $N \in \mathbb{N}^d$. Then for arbitrary $q > 0$*

$$(I) \qquad \sum_N \frac{g(\mathbf{1}, N)^\alpha}{\langle N \rangle^{1+\frac{r}{q}}} < +\infty$$

implies

$$\lim \frac{S_N}{\langle N \rangle^{\frac{1}{q}}} = 0 \quad a.s.$$

PROOF. Using Remark 5 we get for all $N \in \mathbb{N}^d$ that:

$$\mathbb{E}\left\{\max_{M \le N} |S_M|^r\right\} \le A_{r,\alpha,d}\; g(\mathbf{1}, N)^\alpha.$$

Let us introduce the notation $b_N = \frac{1}{\langle N \rangle^{\frac{r}{q}}}$. Since $\prod_{m=1}^d \left\{ \frac{1}{n_m^{\frac{r}{q}}} - \frac{1}{(n_m+1)^{\frac{r}{q}}} \right\} \le$ $\le C \frac{1}{\langle N \rangle^{1+\frac{r}{q}}}$ for some $C > 0$, so (I) implies

$$\sum_N (-1)^d g(\mathbf{1}, N)^\alpha \Delta b_{N+\mathbf{1}} < +\infty.$$

Finally, we apply Lemma 10 and Theorem 3 to obtain the result. ∎

5. A Brunk–Prokhorov Type Theorem

Let $(\Omega, \mathcal{A}, P)$ be a probability space. Let X_N and $\mathcal{A}_N$ be a d-sequence of random variables and be a d-sequence of σ-subalgebras of $\mathcal{A}$, respectively. We shall say that the pair $(X_N, \mathcal{A}_N)$ has property (ex) if

$$(ex) \qquad \mathbb{E}\left(\mathbb{E}(X_L|\mathcal{A}_M)|\mathcal{A}_N)\right) = \mathbb{E}\left(X_L|\mathcal{A}_{\min(M,N)}\right) \qquad L, M, N \in \mathbb{N}^d.$$

This property is widely used in the theory of multiindex martingales (see e.g. [5]). Let X_N be a d-sequence of random variables and $\mathcal{A}_N$ a nondecreasing d-sequence of sub σ-algebras of $\mathcal{A}$. We say that X_N is a martingale difference if

$$X_N \quad \text{is measurable with respect to} \quad \mathcal{A}_N, \quad N \in \mathbb{N}^d,$$
$$\mathbb{E}(X_\mathbf{1}) = 0 \quad \text{and} \quad \mathbb{E}(X_N|\mathcal{A}_M) = 0 \quad \text{if} \quad M < N.$$

In this section we shall use the Doob and the Burkholder inequalities for d-sequences of random variables. For the sake of completeness we state and prove these inequalities in the lemma below.

LEMMA 12. *(a)* (Doob's L^p-inequality.) *Let $p > 1$. Then for any martingale $(X_N, \mathcal{A}_N)$ having property (ex) for arbitrary $N \in \mathbb{N}^d$*

$$\mathbb{E}\left\{\max_{M \le N} |X_M|^p\right\} \le \left\{\frac{p}{p-1}\right\}^{pd} \mathbb{E}(|X_N|^p).$$

(b) (Burkholder's inequality) *Let $p > 1$. Then there is a constant $D_{p,d}$ such that for any martingale difference X_N having property (ex)*

$$\mathbb{E}(|S_N|^{2p}) \le D_{p,d}\mathbb{E}\left(\left\{\sum_{M \le N} X_M^2\right\}^p\right) \qquad \forall N \in \mathbb{N}^d.$$

PROPOSITION 13. *Let X_N be a martingale difference having property (ex) and $p \ge 1$. Suppose that $\sum_{M \le N} E(|X_M|^{2p}) \le C\langle N\rangle^r$ for some $C > 0$ and $r < p + 1$. Then $\lim_N \frac{S_N}{\langle N\rangle} = 0$ a.s.*

PROOF. From Burkholder's inequality (Lemma 12(b)) and Hölder's inequality

$$\mathbb{E}\left(|S_N|^{2p}\right) \le D_{2p,2}\mathbb{E}\left\{\left\{\sum_{M \le N} X_M^2\right\}^p\right\} \le$$
$$\le D_{2p,2}\langle N\rangle^{p-1} \sum_{M \le N} \mathbb{E}\left(|X_M|^{2p}\right) \le D_{2p,2}\langle N\rangle^{p+r-1}.$$

Thus, by Doob's inequality (Lemma 12(a)),

$$\mathbb{E}\left\{\max_{M \le N} |S_M|^{2p}\right\} \le F_{2p,2} \sum_{M \le N} \Delta\langle M\rangle^{p+r-1}$$

for some constant $F_{2p,2} > 0$. Now $\Delta\langle M\rangle^{p+r-1} \le C\langle M\rangle^{p+r-2}$ and Theorem 3 implies the result. ∎

PROPOSITION 14. *Let X_N be a martingale difference having property (ex) and let $p \ge 1$. Suppose that $\mathbb{E}(|X_N|^{2p})$ is d-sequence of product type. Then*

$$\sum_N \frac{\mathbb{E}(|X_N|^{2p})}{b_N^{2p}} \langle N\rangle^{p-1} < +\infty$$

implies $\lim_N \frac{S_N}{b_N} = 0$ *a.s., provided that* b_N *is a nondecreasing, positive, unbounded* d*-sequence of product type and either* $p = 1$ *or* $\frac{\langle N\rangle^\delta}{b_N}$ *is nonincreasing for some* $\delta > \frac{p-1}{2p}$.

PROOF. Applying Lemma 12(b), Hölder's inequality and Lemma 12(a) we get

$$\mathbb{E}\left\{\max_{M\le N}|S_M|^{2p}\right\} \le C_{p,d}\langle N\rangle^{p-1}\sum_{M\le N}\mathbb{E}(|X_M|^{2p})$$

for some $C_{p,d} > 0$. In case $p = 1$ our main theorem and the above inequality imply the result. Let $p > 1$. Introduce the notation $c_N = \langle N\rangle^{p-1}\sum_{M\le N}\mathbb{E}(|X_M|^{2p})$. It is easy to see that

$$\Delta c_N = \prod_{l=1}^{d}\left\{n_l^{p-1}\sum_{k=1}^{n_l}a_k^{(l)} - (n_l-1)^{p-1}\sum_{k=1}^{n_l-1}a_k^{(l)}\right\} =$$

$$= \prod_{l=1}^{d}\left\{n_l^{p-1}a_{n_l}^{(l)} + \left\{n_l^{p-1} - (n_l-1)^{p-1}\right\}\sum_{k=1}^{n_l-1}a_k^{(l)}\right\} \le$$

$$\le \prod_{l=1}^{d}\left\{n_l^{p-1}a_{n_l}^{(l)} + Cn_l^{p-2}\sum_{k=1}^{n_l-1}a_k^{(l)}\right\}$$

for some $C > 0$, where $\prod_{l=1}^{d}a_{n_l}^{(l)} = \mathbb{E}(|X_N|^{2p})$. Using the assumptions we get

$$\sum_{m=1}^{n}\frac{m^{p-2}}{b_m^{(l)2p}}\sum_{k=1}^{m-1}a_k^{(l)} = \sum_{k=1}^{n-1}a_k^{(l)}\sum_{m=k+1}^{n}\frac{m^{p-2}}{b_m^{(l)2p}} \le$$

$$\le \sum_{k=1}^{n}a_k^{(l)}\sum_{m=k}^{\infty}\frac{m^{p-2}}{b_m^{(l)2p}} == \sum_{k=1}^{n}a_k^{(l)}\sum_{m=k}^{\infty}\frac{1}{m^r}\frac{m^{p+r-2}}{b_m^{(l)2p}} \le$$

$$\le \sum_{k=1}^{n}a_k^{(l)}\frac{k^{p+r-2}}{b_k^{(l)2p}}\sum_{m=k}^{\infty}\frac{1}{m^r} \le \sum_{k=1}^{n}a_k^{(l)}\frac{k^{p+r-2}}{b_k^{(l)2p}}Ck^{1-r}$$

for some $r > 1$, $C_r > 0$ and for each $1 \le l \le d$. This means that $\sum_N \Delta\frac{c_N}{b_N^{2p}} < +\infty$, hence one can apply Theorem 3. ∎

We remark that a similar proposition can be proved in a similar manner for d-sequences having maximal coefficient of correlation strictly smaller than 1. For this, one can use [14, Lemma 4] instead of Burkholder's inequality.

6. Mixingales

In this chapter we define multiindex L^r mixingales and prove an SLLN for a special class of such random variables. Remark that the notion of L^r mixingales was introduced by McLeish [10] and Andrews [1]. Let $\mathbb{Z}$ denote the set of integers and let

$$\mathcal{E}_N = \left\{ M \in \mathbb{Z}^d : 0 \le n_k - m_k \le 1,\ k = 1, \dots, d \text{ and } \textstyle\sum_{k=1}^d (n_k - m_k) \text{ is even} \right\},$$

$$\mathcal{O}_N = \left\{ M \in \mathbb{Z}^d : 0 \le n_k - m_k \le 1,\ k = 1, \dots, d \text{ and } \textstyle\sum_{k=1}^d (n_k - m_k) \text{ is odd} \right\},$$

if $N \in \mathbb{Z}^d$.

DEFINITION 15. Let $r \ge 1$, $(\Omega, \mathcal{A}, P)$ be a probability space, X_N be a d-sequence of random variables with finite r-th moment, $\mathcal{A}_N$ $(N \in \mathbb{Z}^d)$ be a nondecreasing d-sequence of σ-subalgebras of $\mathcal{A}$. The pair $(X_N, \mathcal{A}_M)$ $(N \in \mathbb{N}^d,\ M \in \mathbb{Z}^d)$ is called L^r-mixingale if

$$(a)\ \|\mathbb{E}(X_N|\mathcal{A}_{N-M})\|_r \le c_N \Psi_{-M} \quad \text{if} \quad m_i \ge 0 \quad \text{for some} \quad i = 1, \dots, d,$$

$$(b)\qquad \|X_N - \mathbb{E}(X_N|\mathcal{A}_{N+M})\|_r \le c_N \Psi_M \quad \text{if} \quad M \ge \mathbf{0},$$

where c_N $(N \in \mathbb{N}^d)$, Ψ_N $(N \in \mathbb{Z}^d)$ are d-sequences with $\Psi_N \to 0$ as $n_i \to -\infty$ for some $i = 1, \dots, d$, $\Psi_N \to 0$ as $n_i \to \infty$ for each $i = 1, \dots, d$, and there is a constant $C > 0$ for which

$$\Psi_M \le C \Psi_N$$

for any $N \in \mathbb{Z}^d$ and $M \in \mathcal{E}_N \cup \mathcal{O}_N$.

The following lemma is a straightforward generalization of Lemma 1 and Lemma 2 of [7].

LEMMA 16. *(a) Let $r \ge 2$ and $(X_N, \mathcal{A}_M)$ $(N \in \mathbb{N}^d, M \in \mathbb{Z}^d)$ be an L^r mixingale, having property (ex). Then there exists an $F_{r,d} > 0$ such that*

$$\left\| \max_{M \le N} |S_M| \right\|_r \le F_{r,d} \sum_{K \in \mathbb{Z}^d} \left\{ \sum_{M \le N} \left\| X_M^{(K)} \right\|_r^2 \right\}^{\frac{1}{2}},$$

where $X_M^{(K)} = \Delta\mathbb{E}(X_M|\mathcal{A}_{M-K})$ *and here the difference is taken according to the subscript of* $\mathcal{A}$ *while the subscript of* X *remains fixed.*

(b) Let $r \geq 2$ *and* $(X_N, \mathcal{A}_M)$ $(N \in \mathbb{N}^d, M \in \mathbb{Z}^d)$ *be an* L^r *mixingale, having property* (ex) *such that* $\sum_{K\in\mathbb{Z}^d} \Psi_K < +\infty$. *Then*

$$\left\| \max_{M\leq N} |S_M| \right\|_r \leq C_{r,d} \left\{ \sum_{M\leq N} c_M^2 \right\}^{\frac{1}{2}}$$

for some $C_{r,d}$.

PROOF. (a) Let $N, K \in \mathbb{N}^d$. Then

$$\sum_{-K\leq M\leq K} X_N^{(M)} = \sum_{-K\leq M\leq K} \Delta\mathbb{E}(X_N|\mathcal{A}_{N-M}) =$$

$$= \mathbb{E}(X_N|\mathcal{A}_{N+K}) + \sum_{L\in\mathcal{L}_K^+} \mathbb{E}(X_N|\mathcal{A}_{N+L}) + (-1) \sum_{L\in\mathcal{L}_K^-} \mathbb{E}(X_N|\mathcal{A}_{N+L}),$$

where

$$\mathcal{L}_K^+ = \{L \in \mathbb{Z}^d \ : \ l_i = k_i \text{ if } i \notin I \text{ and } l_i = -(k_i+1) \text{ if } i \in I, \text{ for some } I \subset \{1,\dots,d\}, \text{ with } I \neq \emptyset \text{ and } \operatorname{card}(I) \text{ is even } \},$$

$$\mathcal{L}_K^- = \{L \in \mathbb{Z}^d \ : \ l_i = k_i \text{ if } i \notin I \text{ and } l_i = -(k_i+1) \text{ if } i \in I, \text{ for some } I \subset \{1,\dots,d\}, \text{ with } \operatorname{card}(I) \text{ is odd } \}.$$

By the definition of the L^r-mixingale, one can see that

$$\lim_K \left\{ \sum_{-K\leq M\leq K} X_N^{(M)} - \mathbb{E}(X_N|\mathcal{A}_{N+K}) \right\} = \mathbf{0} \quad \text{in } L^r$$

and so

$$\lim_K \left\{ \sum_{-K\leq M\leq K} X_N^{(M)} - X_N \right\} = \mathbf{0} \quad \text{in } L^r .$$

Hence, using the triangle inequality in L^r, we get

$$\left\| \max_{M\leq N} |S_M| \right\|_r = \left\| \max_{M\leq N} \left| \sum_{L\leq M} \sum_{K\in Z^d} X_L^{(K)} \right| \right\|_r \leq$$

$$\leq \left\| \max_{M \leq N} \sum_{K \in Z^d} \left| \sum_{L \leq M} X_L^{(K)} \right| \right\|_r \leq \sum_{K \in Z^d} \left\| \max_{M \leq N} \left| \sum_{L \leq M} X_L^{(K)} \right| \right\|_r = (I).$$

Let $K \in \mathbb{N}^d$ be fixed. With the help of property (ex) it is easy to check that the pair $(Z_M, \mathcal{F}_M)$ is martingale difference, where

$$Z_M = X_M^{(K)} \quad \text{and} \quad \mathcal{F}_M = \mathcal{A}_{M-K}.$$

Hence by Lemma 12 (a), (b) and by the triangle inequality in the space $L^{\frac{r}{2}}$, we have

$$(I) \leq D_{r,d} \sum_{K \in Z^d} \left\| \sum_{L \leq N} X_L^{(K)} \right\|_r \leq F_{r,d} \sum_{K \in Z^d} \left\| \left\{ \sum_{L \leq N} \left| X_L^{(K)} \right|^2 \right\}^{\frac{1}{2}} \right\|_r =$$

$$= F_{r,d} \sum_{K \in Z^d} \left\| \left\{ \sum_{L \leq N} \left| X_L^{(K)} \right|^2 \right\} \right\|_{\frac{r}{2}}^{\frac{1}{2}} \leq F_{r,d} \sum_{K \in Z^d} \left\{ \sum_{L \leq N} \left\| \left| X_L^{(K)} \right|^2 \right\|_{\frac{r}{2}} \right\}^{\frac{1}{2}}.$$

(b) Let us consider $X_L^{(K)}$. If $k_m \geq 0$ for some $m = 1, \ldots, d$ then

$$\|\Delta\mathbb{E}(X_L|\mathcal{A}_{L-K})\|_r \leq c_L 2^d C \Psi_{-K}.$$

Otherwise, if $k_m \leq -1$ for each $m = 1, \ldots, d$, then by Definition 15,

$$\|\Delta\mathbb{E}(X_L|\mathcal{A}_{L-K})\|_r \leq \sum_{M \in \mathcal{E}_{L-K} \cup \mathcal{O}_{L-K}} \|X_L - \mathbb{E}(X_L|\mathcal{A}_M)\|_r \leq c_L 2^d C \Psi_{-K}.$$

Hence, by part (a),

$$\left\| \max_{M \leq N} |S_M| \right\|_r \leq F_{r,d} \sum_{K \in Z^d} \left\{ \sum_{L \leq N} c_L^2 2^{2d} C^2 \Psi_{-K}^2 \right\}^{\frac{1}{2}} =$$

$$= F_{r,d} 2^d C \left\{ \sum_{K \in Z^d} \Psi_K \right\} \left\{ \sum_{L \leq N} c_L^2 \right\}^{\frac{1}{2}}. \qquad \blacksquare$$

PROPOSITION 17. *Let $r \geq 2$ and $(X_N, \mathcal{A}_M)$ $(N \in \mathbb{N}^d, M \in \mathbb{Z}^d)$ be an L^r mixingale of property (ex). Then*

$$\sum_{N \in \mathbb{Z}^d} \Psi_N < \infty \quad \text{and} \quad \sum_{N \in \mathbb{N}^d} \frac{1}{\langle N \rangle^{1+\frac{r}{q}}} \left\{ \sum_{M \leq N} c_M^2 \right\}^{\frac{r}{2}} < \infty$$

imply

$$\lim_N \frac{S_N}{\langle N \rangle^{\frac{1}{q}}} = 0 \quad a.s.$$

provided that the d-sequence c_N is of product type.

PROOF. Easy consequence of Proposition 11 and Lemma 16(b). ∎

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