Ukrainian Mathematical Journal, Vol. 52, No. 2, 2000

ON THE ROSENTHAL INEQUALITY FOR MIXING FIELDS

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UDC 519.21

A proof of the Rosenthal inequality for α-mixing random fields is given. The statements and proofs are modifications of the corresponding results obtained by Doukhan and Utev.

1. Introduction and Results

The Rosenthal inequalities are important tools to prove the consistency of certain estimators for weakly dependent random processes and fields (see, e.g., [1]). The first version of such inequalities was proved by Rosenthal [2] for independent random variables. The Rosenthal inequalities for mixing sequences were obtained by Utev [3] and for mixing fields by Doukhan [4]. However, Doukhan noted that the proof of the interpolation lemma in [3] is "not clear" (see [4, p. 27]). Actually, the first inequality in the expression preceding (4.4) in [3] seems to be not valid. Therefore, one cannot use Lemma 4.1 from [3], and, thus, the extension of the Rosenthal inequality from positive even integer exponents to arbitrary positive real exponents is an open problem. On the other hand, Doukhan [4] presented the Rosenthal inequalities for α-mixing and φ-mixing fields. However, in the opinion of the authors of the present paper, there is a gap in the proof of Theorem 1 in [4, p. 29].

The aim of the present paper is to give a version of the Rosenthal inequality for α-mixing fields. The results and proofs presented here are slight modifications of the corresponding results presented in [4] and [3]. The authors want to summarize what is clear in the abovementioned papers concerning the topic. Similar considerations can be made in the φ-mixing case (see also Remark 4 in [4, p. 32]).

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Random variables are supposed to be defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{A}$ and $\mathcal{B}$ be two σ-algebras in $\mathcal{F}$. The α-mixing coefficient is defined as follows:

$$\alpha(\mathcal{A}, \mathcal{B}) = \sup\{|\mathbb{P}(A)\mathbb{P}(B) - \mathbb{P}(AB)| : A \in \mathcal{A},\ B \in \mathcal{B}\}.$$

The covariance inequality in the α-mixing case is the following (see, e.g., [4, p. 9]):

$$|\operatorname{cov}(X, Y)| \le 8\,[\alpha(\sigma(X), \sigma(Y))]^{1/r}\|X\|_p\|Y\|_q,$$

$$r, p, q \ge 1, \qquad \frac{1}{r} + \frac{1}{p} + \frac{1}{q} = 1.$$

Let I be the set of integer lattice points in $\mathbb{R}^d$, $d \ge 1$. The space $\mathbb{R}^d$ will be considered with the maximum norm and the distance generated by this norm. Let $\{Y_t : t \in I\}$ be a set of random variables. The α-mixing coefficient of Y is

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Published in Ukrainskii Matematicheskii Zhurnal, Vol. 52, No. 2, pp. 266–276, February, 2000. Original article submitted March 26, 1998.

0041-5995/00/5202-0305 $25.00

$$\alpha_Y(r,u,v) = \sup\{\alpha(\mathcal{F}_{I_1},\mathcal{F}_{I_2})\colon \text{distance}(I_1,I_2) \ge r,\ \text{card}(I_1) \le u,\ \text{card}(I_2) \le v\},$$

where I_1 and I_2 are finite subsets in I and $\mathcal{F}_{I_i} = \sigma\{Y_t\colon t \in I_i\}$, $i = 1, 2$.

Let T be a finite set in I. We introduce the following notation:

$$L(\mu,\varepsilon,T) = \sum_{t\in T} (\mathbb{E}|Y_t|^{\mu+\varepsilon})^{\mu/(\mu+\varepsilon)} = \sum_{t\in T} \|Y_t\|^{\mu}_{\mu+\varepsilon},$$

$$D(h,\varepsilon,T) = \begin{cases} L(h,0,T) & \text{if } 0<h\le 1,\ \varepsilon\ge 0,\\ L(h,\varepsilon,T) & \text{if } 1<h\le 2,\ \varepsilon\ge 0,\\ \max\{L(h,\varepsilon,T),[L(2,\varepsilon,T)]^{h/2}\} & \text{if } 2<h,\ \varepsilon\ge 0.\end{cases}$$

Let s_r and b_r denote the number of points of I in a sphere with radius r and center in I and in a ball with radius r and center in I, respectively: $s_r = \text{card}(\{t\colon \|t\| = r\}\cap I)$ and $b_r = \text{card}(\{t\colon \|t\| \le r\}\cap I)$. Let

$$c^{(\alpha)}_{u,h-u} = 8u!\,(h-u-1)!\,(h-1)! \sum_{r=1}^{\infty} [\alpha_Y(r,u,h-u)]^{\varepsilon/(h+\varepsilon)} s_r b_r^{h-2}.$$

The following theorem is a version of Theorem 1 in [4, p. 26]. The assumptions here are stronger than those in [4]. The explicit formulas for the constants are given.

Theorem 1. *Let $l>1$ and $\varepsilon>0$. Let Y_t, $t\in I$, be centered random variables with $\mathbb{E}|Y_t|^{l+\varepsilon}<\infty$, $t\in I$. Let h be the smallest even integer with $h\ge l$. Assume that $c^{(\alpha)}_{u,h-u}<\infty$ for $u=1,\dots,h-1$. Then there exists a constant $K_{(\alpha)}$ such that*

$$\mathbb{E}\left|\sum_{t\in T} Y_t\right|^l \le K_{(\alpha)} D(l,\varepsilon,T) \tag{1}$$

for any finite subset T of I.

Remark 1. $K_{(\alpha)}$ does not depend on T but it depends on the mixing coefficients and l, namely, $K_{(\alpha)} = H^{(\alpha)}_h C_l$, where

$$H^{(\alpha)}_h = 1 + \sum_{u=1}^{h-1} c^{(\alpha)}_{u,h-u} + \sum_{u=2}^{h-2} \binom{h}{u} H^{(\alpha)}_u H^{(\alpha)}_{h-u},$$

$$C_l = 2^{(h-l+\varepsilon)(2h+2l-1)/\varepsilon};$$

here, we assume that $0<\varepsilon<l/2$. If l is an even integer, then one can set $C_l = 1$.

Remark 2. Inequality (1) is always satisfied for $0<l\le 1$ if we replace $K_{(\alpha)}$ by 1.

Remark 3. The above result is valid in the following, slightly more general, setting: If I is a regular pattern in $\mathbb{R}^d$, then s_r should be replaced by $\tilde{s}_r = \operatorname{card}(\{t: r-1 < \|t\| \le r\} \cap I)$, i.e., $\tilde{s}_r$ denotes the number of points of I in a ring with radius r, thickness 1, and center in I.

Remark 4. For the case $d = 1$, i.e., for mixing sequences, see [4, p. 26].

2. Auxiliary Results and Interpolation Lemma

Lemma 1. *Let L be a finite subset in a metric space (M, ρ). Suppose that the minimal distance of two nonempty complementary subsets of L is r. Then one can choose two nonempty complementary subsets A and B in L such that the distance between A and B is r and there exists a connected graph with edges not longer than r and with the set of vertices A; the same is true for B.*

Proof. Let $s, t \in U \subseteq L$. We say that s is r-connected with t in U if there exists a connected graph with edges not longer than r and with vertices in U and, moreover, s and t are vertices of this graph. Let S_1 and S_2 be two nonempty complementary subsets of L such that $\rho(S_1, S_2) = r$. Consider points $t_1 \in S$ and $t_2 \in S_2$ such that $\rho(t_1, t_2) = r$. Let $S_i^{(1)} \subseteq S_i$ be the set of points r-connected with t_i in S_i, $i = 1, 2$. We have

$$\rho\left(\{S_1^{(1)} \cup S_2^{(1)}\}, \{(S_1 - S_1^{(1)}) \cup (S_2 - S_2^{(1)})\}\right) \ge r.$$

But r is the maximal distance between the subsets of L and, therefore, either the second subset is empty or the distance is r. In the first case, we are done. In the second case, let $\tilde{S}_1^{(1)} \subseteq S_1 - S_1^{(1)}$ be the set of points r-connected with $S_2^{(1)}$ in $(S_1 - S_1^{(1)}) \cup S_2^{(1)}$. The definition of $\tilde{S}_2^{(1)}$ is similar. Obviously, $\tilde{S}_1^{(1)} \cup \tilde{S}_2^{(1)} \ne \varnothing$. We now consider $(S_1 - \tilde{S}_1^{(1)}) \cup \tilde{S}_2^{(1)}$ and $(S_2 - \tilde{S}_2^{(1)}) \cup \tilde{S}_1^{(1)}$. The distance between these two sets is r. Moreover, in these sets, the number of points r-connected with t_1 in $(S_1 - \tilde{S}_1^{(1)}) \cup \tilde{S}_2^{(1)}$ or the number of points r-connected with t_2 in $(S_2 - \tilde{S}_2^{(1)}) \cup \tilde{S}_1^{(1)}$ is greater than at the starting situation. Repeating the above procedure, we obtain the required result.

The following lemma is a version of Lemma 2 in [4, p. 29], where it was stated for even integer $(a + b)$ such that $(a + b) \ge 2$.

Lemma 2. *If $\delta \ge 0$, $a \ge 2$, and $b \ge 2$ are real numbers, then*

$$D(a, \delta, T)\, D(b, \delta, T) \le D(a + b, \delta, T).$$

The proof will be based on the Hölder inequality:

1. Let X and Y be real random variables. If $p > 1$ and $q = p/(p-1)$, then

$$\mathbb{E}\,|XY| \le \|X\|_p \|Y\|_q. \tag{2}$$

2. If $a_i, b_i \in \mathbb{R}$ $(i = 1, \dots, n)$, $p > 1$, and $q = p/(p-1)$, then

$$\sum_{i=1}^{n} |a_i b_i| \le \left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |b_i|^q\right)^{1/q}. \tag{3}$$

Proof. We set

$$L_\nu = L(\nu, \delta, T),$$

$$D_\nu = D(\nu, \delta, T),$$

$$X_t = Y_t L_2^{-1/2} \quad \text{for} \quad t \in T,$$

$$L_\nu^* = \sum_{t \in T} \|X_t\|_{\nu+\delta}^\nu,$$

$$D_\nu^* = L_\nu^* \vee (L_2^*)^{\nu/2} \quad \text{if} \quad \nu \geq 2,$$

$$c = a + b.$$

Then

$$L_\nu^* = \sum_{t \in T} \left(\mathbb{E}|Y_t L_2^{-1/2}|^{\nu+\delta}\right)^{\nu/(\nu+\delta)} = L_2^{-\nu/2} L_\nu.$$

Thus, we get

$$D_\nu^* = L_2^{-\nu/2} L_\nu \vee L_2^{-\nu/2} L_2^{\nu/2} = L_2^{-\nu/2} D_\nu \quad \text{for} \quad \nu \geq 2, \tag{4}$$

and

$$L_2^* = 1. \tag{5}$$

By using (5), we obtain

$$D_\nu^* = L_\nu^* \vee (L_2^*)^{\nu/2} = L_\nu^* \vee 1 \quad \text{for} \quad \nu \geq 2. \tag{6}$$

For any $a \geq 2$ and $b \geq 2$, this equality yields

$$D_a^* D_b^* = L_a^* L_b^* \vee L_a^* \vee L_b^* \vee 1. \tag{7}$$

(a) First, we assume that $a > 2$. We set

$$u = \frac{(c+\delta)(a-2)}{c-2} \quad \text{and} \quad v = \frac{(2+\delta)(c-a)}{c-2}.$$

Then $u + v = a + \delta$ and, hence, using (2) with $p = \dfrac{c+\delta}{u}$ and $q = \dfrac{2+\delta}{v}$, we obtain

$$\mathbb{E}|X_t|^{a+\delta} = \mathbb{E}|X_t|^{u+v} \leq \left\||X_t|^u\right\|_{(c+\delta)/u} \left\||X_t|^v\right\|_{(2+\delta)/v}.$$

This inequality yields

$$L_a^* \le \sum_{t\in T} \|X_t\|_{c+\delta}^{rc} \|X_t\|_{2+\delta}^{2s}, \tag{8}$$

where

$$r = \frac{ua}{c(a+\delta)}, \quad s = \frac{av}{2(a+\delta)}.$$

Since $0 < r < a/c < 1$, by using (3) with $p = 1/r$ and $q = 1/(1-r)$ we obtain from (8) that $L_a^* \le (L_c^*)^r A^{1-r}$, where

$$A = \sum_{t\in T} \|X_t\|_{2+\delta}^{2s/(1-r)}.$$

Since $s/(1-r) \ge 1$, it follows from (5) that $A \le 1$ and, therefore, $L_a^* \le (L_c^*)^r$. Hence, if $L_a^* \ge 1$, then $L_c^* \ge 1$. Therefore, since $0 < r < a/c < 1$, we get

$$L_a^* \le (L_c^*)^r \le (L_c^*)^{a/c} \le L_c^* \quad \text{if} \quad L_a^* \ge 1. \tag{9}$$

(a′) We now concentrate on the case where $a > 2$ and $b > 2$. Then relation (9) is valid for b, namely,

$$L_b^* \le (L_c^*)^{b/c} \le L_c^* \quad \text{if} \quad L_b^* \ge 1. \tag{10}$$

These inequalities yield $L_b^* L_b^* \le L_c^* \vee 1$. Therefore, using (7), (9), (10), and (6), we obtain

$$D_a^* D_b^* \le (L_c^* \vee 1) \vee L_a^* \vee L_b^* = L_c^* \vee 1 = D_c^*.$$

Hence, using (4), we get the required statement.

(b) We now assume that $a = b = 2$. Then, by using (7), (5), and (6), we get

$$D_2^* D_2^* = 1 \le 1 \vee L_4^* = D_4^*.$$

Hence, using (4), we obtain the required statement.

(c) If $a > 2$ and $b = 2$, then relations (5), (6), and (9) yield

$$D_a^* D_2^* = D_a^* \le D_c^*.$$

Hence, using (4), we obtain the required statement.

(d) Finally, if $b > 2$ and $a = 2$, then the proof is the same as in case (c).

This completes the proof of Lemma 2.

The interpolation lemma presented below is a version of Lemma 4.4 in [3] and Lemma 1 in [4, p. 27].

Let B be a separable Banach space with norm $\| \ \|$. Let $F = \{\mathcal{F}_1, \dots, \mathcal{F}_n\}$ be a family of sub-σ-algebras of the σ-algebra $\mathcal{F}$ and let $\eta = \{\eta_1, \dots, \eta_n\}$ be a family of centered random variables. The family η is called (F, B)-adapted if η_i is B-valued and $\mathcal{F}_i$-measurable. We shall use the following notation:

$$M(\nu, \delta, \eta) = \sum_{i=1}^{n} \left(\mathbb{E}\|\eta_i\|^{\nu+\delta}\right)^{\nu/(\nu+\delta)} = \sum_{i=1}^{n} \|\eta_i\|_{\nu+\delta}^{\nu},$$

$$Q(\nu, \delta, \eta) = \begin{cases} M(\nu, \delta, \eta) & \text{if } 1 \le \nu \le 2, \\ M(\nu, \delta, \eta) \vee M^{\nu/2}(2, \delta, \eta) & \text{if } \nu > 2, \end{cases}$$

where $a \vee b = \max\{a, b\}$. $\mathrm{I}\{A\}$ denotes the indicator function of the set A.

Lemma 3. *Assume that, for some fixed real constants $\nu \ge 1$, $\delta > 0$, and $c \ge 1$, any (F, B)-adapted centered family $\eta = \{\eta_1, \dots, \eta_n\}$ satisfies the inequality*

$$\mathbb{E}\left\|\sum_{i=1}^{n} \eta_i\right\|^{\nu} \le c\,Q(\nu, \delta, \eta). \tag{11}$$

We set $t_0 = 1 \vee (\nu/2) \vee (\nu - \delta)$. Then, for any t with $t_0 \le t \le \nu$ and any (F, B)-adapted centered family $\varphi = \{\varphi_1, \dots, \varphi_n\}$, we have

$$\mathbb{E}\left\|\sum_{i=1}^{n} \varphi_i\right\|^{t} \le c\,2^{4\nu-1} Q(t, \delta, \varphi).$$

Note that $c \ge 1$ is a consequence of the other assumptions. In order to prove the lemma, we require the following known inequalities:

1. C_p-inequality. If $x, y \in B$ and $p \ge 1$, then

$$\|x+y\|^p \le 2^{p-1}(\|x\|^p + \|y\|^p); \tag{12}$$

if $0 < p \le 1$, then

$$\|x+y\|^p \le \|x\|^p + \|y\|^p. \tag{13}$$

2. Let X be a B-valued random variable. If $p \ge 1$, then

$$(\mathbb{E}\,\|X\|)^p \le \mathbb{E}(\|X\|^p) \tag{14}$$

and

$$\mathbb{E}\,\|X - \mathbb{E}X\|^p \le 2^p\mathbb{E}\,\|X\|^p. \tag{15}$$

If $0 < p \le 1$, then

$$(\mathbb{E}\ \|X\|)^p \geq \mathbb{E}\ \|X\|^p \tag{16}$$

and

$$\mathbb{E}\,\|X - \mathbb{E}X\|^p \leq 2(\mathbb{E}\ \|X\|)^p, \tag{17}$$

where $\mathbb{E}\,X$ is the Bochner integral.

3. If X is a B-valued random variable and $0 < q \leq p$, then

$$\|X\|_q \leq \|X\|_p \tag{18}$$

where $\|X\|_q = (\mathbb{E}\ \|X\|^q)^{1/q}$.

4. If $a_i \in \mathbb{R}$ $(i = 1, \dots, n)$ and $p \geq 1$, then

$$\sum_{i=1}^{n} |a_i|^p \leq \left(\sum_{i=1}^{n} |a_i|\right)^p. \tag{19}$$

Proof of Lemma 3. Let $\varphi = \{\varphi_1, \dots, \varphi_n\}$ be a centered (F, B)-adapted family of random variables and let $t_0 \leq t \leq \nu$ be a fixed constant. We set

$$Q = Q(t, \delta, \varphi),$$

$$y = Q^{1/t},$$

$$T_i = \varphi_i \mathrm{I}\{\|\varphi_i\| \leq y\}, \quad i = 1, \dots, n,$$

$$Y_i = \varphi_i \mathrm{I}\{\|\varphi_i\| > y\}, \quad i = 1, \dots, n,$$

$$\eta_i = Y_i - \mathbb{E}\,Y_i, \quad i = 1, \dots, n,$$

$$\eta = \{\eta_1, \dots, \eta_n\},$$

$$\psi_i = T_i - \mathbb{E}\,T_i, \quad i = 1, \dots, n,$$

$$\psi = \{\psi_1, \dots, \psi_n\},$$

$$\xi_i = z(\|\eta_i\|^{t/\nu} - \mathbb{E}\ \|\eta_i\|^{t/\nu}), \quad \text{where } z \in B, \quad \|z\| = 1, \quad i = 1, \dots, n,$$

$$\xi = \{\xi_1, \dots, \xi_n\}.$$

Then $\eta_i + \psi_i = \varphi_i$ and $t \geq 1$ and, hence, relation (12) yields

$$\mathbb{E}\left\|\sum_{i=1}^{n}\varphi_i\right\|^t \le 2^{t-1}\left(\mathbb{E}\left\|\sum_{i=1}^{n}\eta_i\right\|^t + \mathbb{E}\left\|\sum_{i=1}^{n}\psi_i\right\|^t\right). \tag{20}$$

Since $\frac{t}{\nu} \le 1$ and $\nu \ge 1$, relations (13) and (12) yield

$$\begin{aligned}\mathbb{E}\left\|\sum_{i=1}^{n}\eta_i\right\|^t &= \mathbb{E}\left(\left\|\sum_{i=1}^{n}\eta_i\right\|^{t/\nu}\right)^{\nu} \\ &\le \mathbb{E}\left(\sum_{i=1}^{n}\|\eta_i\|^{t/\nu}\right)^{\nu} = \mathbb{E}\left(\sum_{i=1}^{n}(\|\eta_i\|^{t/\nu} - \mathbb{E}\|\eta_i\|^{t/\nu}) + \sum_{i=1}^{n}\mathbb{E}\|\eta_i\|^{t/\nu}\right)^{\nu} \\ &\le 2^{\nu-1}\left(\mathbb{E}\left\|\sum_{i=1}^{n}\xi_i\right\|^{\nu} + \left(\sum_{i=1}^{n}\mathbb{E}\|\eta_i\|^{t/\nu}\right)^{\nu}\right).\end{aligned}$$

We set

$$V = cQ(\nu,\delta,\xi) \quad \text{and} \quad W = \left(\sum_{i=1}^{n}\mathbb{E}\|\eta_i\|^{t/\nu}\right)^{\nu}.$$

Since ξ is centered and (F,B)-adapted, the last inequality and (11) yield

$$\mathbb{E}\left\|\sum_{i=1}^{n}\eta_i\right\|^t \le 2^{\nu-1}(V+W). \tag{21}$$

Since ψ is centered and (F,B)-adapted and $\nu/t \ge 1$, by using (14) and (11) we get

$$\mathbb{E}\left\|\sum_{i=1}^{n}\psi_i\right\|^t \le \left(\mathbb{E}\left\|\sum_{i=1}^{n}\psi_i\right\|^{\nu}\right)^{t/\nu} \le U, \tag{22}$$

where $U = (cQ(\nu,\delta,\psi))^{t/\nu}$. Then (20), (21), and (22) yield

$$\mathbb{E}\left\|\sum_{i=1}^{n}\varphi_i\right\|^t \le 2^{t-1}U + 2^{t+\nu-2}V + 2^{t+\nu-2}W. \tag{23}$$

We thus have to estimate the terms U, V, and W.

(U) We set $u = \nu(t+\delta)/t(\nu+\delta)$. Then $u \ge 1$ and, furthermore, $\nu+\delta \ge 1$. Hence, relations (15) and (14) yield

$$\mathbb{E}\|\psi_i\|^{\nu+\delta} \le 2^{\nu+\delta}\mathbb{E}\|T_i\|^{\nu+\delta} \le 2^{\nu+\delta}(\mathbb{E}\|T_i\|^{u(\nu+\delta)})^{1/u}.$$

Thus,

$$M(\nu, \delta, \psi) \le \sum_{i=1}^{n} \left(2^{\nu+\delta}(\mathbb{E}\|T_i\|^{u(\nu+\delta)})^{1/u}\right)^{\nu/(\nu+\delta)} = 2^{\nu} \sum_{i=1}^{n} \left(\mathbb{E}\|T_i\|^{u(\nu+\delta)}\right)^{\nu/u(\nu+\delta)}. \tag{24}$$

We have

$$\mathrm{I}\{\|\varphi_i\| \le y\}\|\varphi_i\|^{u(\nu+\delta)-(t+\delta)} \le y^{u(\nu+\delta)-(t+\delta)}$$

because $u(\nu+\delta) - (t+\delta) \ge 0$. Hence,

$$\mathbb{E}\|T_i\|^{u(\nu+\delta)} = \mathbb{E}\left(\|\varphi_i\|^{t+\delta}\mathrm{I}\{\|\varphi_i\| \le y\}\|\varphi_i\|^{u(\nu+\delta)-(t+\delta)}\right) \le Q^{(t+\delta)(\nu/t-1)/t}\,\mathbb{E}\|\varphi_i\|^{t+\delta}.$$

Using this inequality and (24), we get

$$M(\nu, \delta, \psi) \le 2^{\nu} Q^{\nu/t-1} M(t, \delta, \varphi) \le 2^{\nu} Q^{\nu/t}. \tag{25}$$

(a) Assume that $\nu \le 2$. Then relation (25) yields

$$Q(\nu, \delta, \psi) = M(\nu, \delta, \psi) \le 2^{\nu} Q^{\nu/t}.$$

(b) If $t \le 2 \le \nu$, then relation (25) yields

$$M^{\nu/2}(2, \delta, \psi) \le (2^2 Q^{2/t})^{\nu/2} = 2^{\nu} Q^{\nu/t},$$

whence

$$Q(\nu, \delta, \psi) = \max \begin{cases} M(\nu, \delta, \psi) \le 2^{\nu} Q^{\nu/t}, \\ M^{\nu/2}(2, \delta, \psi) \le 2^{\nu} Q^{\nu/t}. \end{cases}$$

(c) Assume that $2 \le t$. Then, by virtue of (15) and the inequality $\|T_i\| \le \|\varphi_i\|$, we have

$$M(2, \delta, \psi) = \sum_{i=1}^{n} \left(\mathbb{E}\|T_i - \mathbb{E}T_i\|^{2+\delta}\right)^{2/(2+\delta)} \le 4 \sum_{i=1}^{n} \left(\mathbb{E}\|T_i\|^{2+\delta}\right)^{2/(2+\delta)} \le 4 M(2, \delta, \varphi) \le 4 Q^{2/t}.$$

This inequality and (25) yield

$$Q(\nu, \delta, \psi) = \max \begin{cases} M(\nu, \delta, \psi) \le 2^{\nu} Q^{\nu/t}, \\ M^{\nu/2}(2, \delta, \psi) \le (4Q^{2/t})^{\nu/2} = 2^{\nu} Q^{\nu/t}. \end{cases}$$

Cases (a), (b), and (c) imply that

$$Q(\nu, \delta, \psi) \le 2^{\nu} Q^{\nu/t}$$

for every $1 \le t \le \nu$, whence

$$U \le (c 2^{\nu} Q^{\nu/t})^{t/\nu} = c^{t/\nu} 2^t Q. \tag{26}$$

(V) Inequality (15) yields

$$\mathbb{E}\,\|\xi_i\|^{\nu+\delta} = \mathbb{E}\left|\|\eta_i\|^{t/\nu} - \mathbb{E}\,\|\eta_i\|^{t/\nu}\right|^{\nu+\delta} \le 2^{\nu+\delta}\mathbb{E}\,\|\eta_i\|^{t(\nu+\delta)/\nu}. \tag{27}$$

This inequality, (18), (15), and $\|Y_i\| \le \|\varphi_i\|$ yield

$$M(\nu,\delta,\xi) \le 2^{\nu}\sum_{i=1}^{n}\|\eta_i\|^t_{t(\nu+\delta)/\nu} \le 2^{\nu}\sum_{i=1}^{n}\|\eta_i\|^t_{t+\delta}$$

$$\le 2^{\nu+t}\sum_{i=1}^{n}(\mathbb{E}\|Y_i\|^{t+\delta})^{t/(t+\delta)} \le 2^{\nu+t}M(t,\delta,\varphi) \le 2^{\nu+t}Q. \tag{28}$$

(a) Assume that $\nu \le 2$. Then (28) yields

$$Q(\nu,\delta,\xi) = M(\nu,\delta,\xi) \le 2^{\nu+t}Q.$$

(b) Assume that $t \le 2 \le \nu$. By virtue of (27) and (15), we have

$$M(2,\delta,\xi) \le 4\sum_{i=1}^{n}(\mathbb{E}\|\eta_i\|^{t(2+\delta)/\nu})^{2/(2+\delta)} \le 4^{1+t/\nu}\sum_{i=1}^{n}(\mathbb{E}\|Y_i\|^{t(2+\delta)/\nu})^{2/(2+\delta)}, \tag{29}$$

where we have used the fact that $t \ge \nu/2$. We now have

$$\mathrm{I}\{\|\varphi_i\| > y\}\|\varphi_i\|^{t(2+\delta)/\nu-(t+\delta)} \le y^{t(2+\delta)/\nu-(t+\delta)}$$

because $t(2+\delta)/\nu - (t+\delta) \le 0$. Therefore, relation (29) yields

$$M(2,\delta,\xi) \le 4^{1+t/\nu}Q^{2((2+\delta)/\nu-(t+\delta)/t)/(2+\delta)}\sum_{i=1}^{n}\left((\mathbb{E}\|\varphi_i\|^{t+\delta})^{t/(t+\delta)}\right)^{2(t+\delta)/t(2+\delta)}.$$

Hence, by using (19), we get

$$M(2,\delta,\xi) \le 4^{1+t/\nu}Q^{2((2+\delta)/\nu-(t+\delta)/t)/(2+\delta)}(M(t,\delta,\varphi))^{2(t+\delta)/t(2+\delta)} \le 4^{1+t/\nu}Q^{2/\nu}.$$

By using this inequality and (28), we obtain

$$Q(\nu,\delta,\xi) = \max\begin{cases} M(\nu,\delta,\xi) \le 2^{\nu+t}Q, \\ M^{\nu/2}(2,\delta,\xi) \le (4^{1+t/\nu}Q^{2/\nu})^{\nu/2} = 2^{\nu+t}Q. \end{cases}$$

(c) Assume that $2 \le t$. Note that (29) is valid in this case and, hence,

$$M(2, \delta, \xi) \le 4^{1+t/\nu} Q^{2/\nu - 2/t} M(2, \delta, \varphi) \le 4^{1+t/\nu} Q^{2/\nu-2/t} Q^{2/t} = 4^{1+t/\nu} Q^{2/\nu}.$$

(Here, we have used the fact that $\mathrm{I}\{\|\varphi_i\| > y\}\|\varphi_i\|^{t(2+\delta)/\nu-(2+\delta)} \le y^{t(2+\delta)/\nu-(2+\delta)}$ and the definition of Q.) By using the previous inequality and (28), we get

$$Q(\nu, \delta, \xi) = \max \begin{cases} M(\nu, \delta, \xi) \le 2^{\nu+t} Q, \\ M^{\nu/2}(2, \delta, \xi) \le (4^{1+t/\nu} Q^{2/\nu})^{\nu/2} = 2^{\nu+t} Q. \end{cases}$$

Cases (a), (b) and (c) imply that

$$Q(\nu, \delta, \xi) \le 2^{\nu+t} Q$$

for every $1 \le t \le \nu$, whence

$$V \le c 2^{\nu+t} Q. \tag{30}$$

(W) By using (16) and (17), we get

$$\sum_{i=1}^{n} \mathbb{E}\|\eta_i\|^{t/\nu} \le \sum_{i=1}^{n} (\mathbb{E}\|Y_i - \mathbb{E}Y_i\|)^{t/\nu} \le 2^{t/\nu} \sum_{i=1}^{n} (\mathbb{E}\|Y_i\|)^{t/\nu}.$$

Furthermore, we have $\mathrm{I}\{\|\varphi_i\| > y\}\|\varphi_i\|^{1-\nu} \le y^{1-\nu}$, whence

$$\sum_{i=1}^{n} \mathbb{E}\|\eta_i\|^{t/\nu} \le 2^{t/\nu} Q^{1/\nu - 1} \sum_{i=1}^{n} \|\varphi_i\|_\nu^t.$$

Since $t + \delta \ge \nu$, we can apply (18). As a result, we get

$$\sum_{i=1}^{n} \mathbb{E}\|\eta_i\|^{t/\nu} \le 2^{t/\nu} Q^{1/\nu-1} M(t, \delta, \varphi) \le 2^{t/\nu} Q^{1/\nu}.$$

Thus,

$$W \le 2^t Q. \tag{31}$$

Finally, relations (23), (26), (30), and (31) yield

$$\mathbb{E}\left\|\sum_{i=1}^{n} \varphi_i\right\|^t \le 2^{t-1}(c^{t/\nu} 2^t Q + 2^{\nu-1} c 2^{\nu+t} Q + 2^{\nu-1} 2^t Q) \le c 2^{4\nu-1} Q.$$

This completes the proof of Lemma 3.

Corollary 1. *Assume that, for some fixed real constants* $\nu \geq 1$, $\delta > 0$, *and* $c \geq 1$ *and any* (F, B)*-adapted centered family* $\eta = \{\eta_1, \ldots, \eta_n\}$, *relation (11) is satisfied. Then, for any* t *such that* $1 \leq t \leq \nu$ *and any* (F, B)*-adapted centered family* $\varphi = \{\varphi_1, \ldots, \varphi_n\}$, *we have*

$$\mathbb{E}\left\|\sum_{i=1}^{n} \varphi_i\right\|^t \leq C Q(t, \delta, \varphi),$$

where $C = c\, 2^{(\nu - t + \delta)(2\nu + 2t - 1)/\delta}$ *if* $t \geq 2\delta$.

Proof. According to Lemma 3, we can decrease the exponent in each step by δ.

3. Proof of Theorem 1

Lemma 4. *Let* T *be a finite subset in* I, *let* h *be a fixed positive integer, and let* $\varepsilon > 0$. *Let* Y_t, $t \in T$, *be centered random variables such that* $\mathbb{E}\,|Y_t|^{h+\varepsilon} < \infty$, $t \in T$. *Let*

$$A_h(T) = \sum_{\tau \in T^h} \left|\mathbb{E}(Y_{t_1} \ldots Y_{t_h})\right|,$$

where $\tau = \{t_1, \ldots, t_h\} \in T^h$. *Then*

$$A_h(T) \leq H_h^{(\alpha)} D(h, \varepsilon, T). \tag{32}$$

Proof. We omit the superscript (α). We shall prove that, for any positive integer h, we have

$$A_h(T) \leq \left(1 + \sum_{u=1}^{h-1} c_{u,h-u}\right) L(h, \varepsilon, T) + \sum_{u=2}^{h-2} \binom{h}{u} A_u(T) A_{h-u}(T). \tag{33}$$

Here,

$$\sum_{u=1}^{h-1} (\cdot) = 0$$

for $h = 1$ and

$$\sum_{u=2}^{h-2} (\cdot) = 0$$

for $h = 1, 2, 3$. The random variables Y_t have expectation zero and, therefore, $A_1(T) = 0$. Moreover, we shall prove

$$A_2(T) \leq (1 + c_{1,1}) L(2, \varepsilon, T). \tag{34}$$

We have

$$A_h(T) \le \sum_{t \in T} |\mathbb{E} Y_t^h| + \sum_{u=1}^{h-1} \sum_{r=1}^{\infty} \sum_{\xi} \sum_{\eta} |\mathbb{E} Y_\xi Y_\eta|, \tag{35}$$

where $\xi = \{t_1, \dots, t_u\} \in T^u$, $\eta = \{t_{u+1}, \dots, t_h\} \in T^{h-u}$, $Y_\xi = Y_{t_1} \dots Y_{t_u}$, $Y_\eta = Y_{t_{u+1}} \dots Y_{t_h}$, and $\sum_\xi \sum_\eta$ denotes summation over all $\xi = \{t_1, \dots, t_u\} \in T^u$ and $\eta = \{t_{u+1}, \dots, t_h\} \in T^{h-u}$ such that the distance between the sets $\{t_1, \dots, t_u\}$ and $\{t_{u+1}, \dots, t_h\}$ is r, i.e., the maximal distance between complementary pairs of nonempty subsets of $\{t_1, \dots, t_h\}$. Note that every $\{t_1, \dots, t_h\} \in T^h$ should appear on the right hand side of (35), i.e., we take into account the order of components of τ. By using the covariance inequality, we get

$$|\mathbb{E} Y_\xi Y_\eta| \le |\mathbb{E} Y_\xi| |\mathbb{E} Y_\eta| + 8[\alpha_Y(r, u, h-u)]^\rho \| Y_\xi \|_\nu \| Y_\eta \|_\mu, \tag{36}$$

where

$$\rho = \frac{\varepsilon}{h+\varepsilon}, \qquad \nu = \frac{h+\varepsilon}{u}, \qquad \mu = \frac{h+\varepsilon}{h-u}.$$

By using the Hölder inequality, we obtain

$$\| Y_\xi \|_\nu = (\mathbb{E}|Y_{t_1} \dots Y_{t_u}|^{(h+\varepsilon)/u})^{u/(h+\varepsilon)} \le \left[\left(\prod_{i=1}^{u} \mathbb{E}|Y_{t_i}|^{h+\varepsilon} \right)^{1/u} \right]^{u/(h+\varepsilon)} = \prod_{i=1}^{u} \|Y_{t_i}\|_{h+\varepsilon}. \tag{37}$$

By virtue of relation (37), the inequality for arithmetic and geometric means, and Lemma 1, we now get

$$\begin{aligned} \sum_\xi \sum_\eta \|Y_\xi\|_\nu \|Y_\eta\|_\mu &\le \sum_\xi \sum_\eta \prod_{i=1}^{u} (\|Y_{t_i}\|_{h+\varepsilon}^h)^{1/h} \prod_{i=u+1}^{h} (\|Y_{t_i}\|_{h+\varepsilon}^h)^{1/h} \\ &\le \frac{1}{h} \sum_\xi \sum_\eta \left(\sum_{i=1}^{u} \|Y_{t_i}\|_{h+\varepsilon}^h + \sum_{i=u+1}^{h} (\|Y_{t_i}\|_{h+\varepsilon}^h \right) \\ &\le \sum_{t \in T} s_r b_r^{h-2} u!(h-u-1)!(h-1)! \|Y_s\|_{h+\varepsilon}^h. \end{aligned} \tag{38}$$

To explain the last inequality, we note that, for any fixed $s \in T$, we can choose the other $u-1$ members of ξ in at most $(u-1)! b_r^{u-1}$ ways, the point closest to η in at most u ways, a point located at a distance r from the point considered in at most s_r ways, and the other $h-u-1$ points in η in at most $(h-u-1)! b_r^{h-u-1}$ ways. Moreover, the factor $(h-1)! = \frac{h!}{h}$ is explained by the different orders of h elements. On the other hand, we have

$$\sum_{r=1}^{\infty} \sum_\xi \sum_\eta |\mathbb{E} Y_\xi| |\mathbb{E} Y_\eta| \le \binom{h}{u} A_u(T) A_{h-u}(T). \tag{39}$$

By virtue of (35), (36), (39), and (38), we now get

$$A_h(T) \leq \sum_{t\in T} |\mathbb{E} Y_t^h| + \sum_{u=1}^{h-1} \binom{h}{u} A_u(T) A_{h-u}(T) + \sum_{u=1}^{h-1} \sum_{s\in T} c_{u,h-u} \|Y_s\|_{h+\varepsilon}^h$$

$$\leq \sum_{u=1}^{h-1} \binom{h}{u} A_u(T) A_{h-u}(T) + \left(1 + \sum_{u=1}^{h-1} c_{u,h-u}\right) L(h, \varepsilon, T),$$

which yields (33). In the simple case $h = 2$, the above arguments give (34). In view of Lemma 2, relation (33) yields (32).

Proof of Theorem 1. If h is an even positive integer, then

$$\mathbb{E}\left(\sum_{t\in T} Y_t\right)^h \leq A_h(T).$$

This and Lemma 4 yield (1) for even l. For arbitrary l, one can use Corollary 1.

This work was partially supported by the Hungarian Ministry of Culture and Education (grant No. FKFP 0429/1997), the International Soros Science Education Program (grant No. APUO 71051), and the Institute of Mathematics and Informatics at the Kossuth University (Debrecen, Hungary).

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