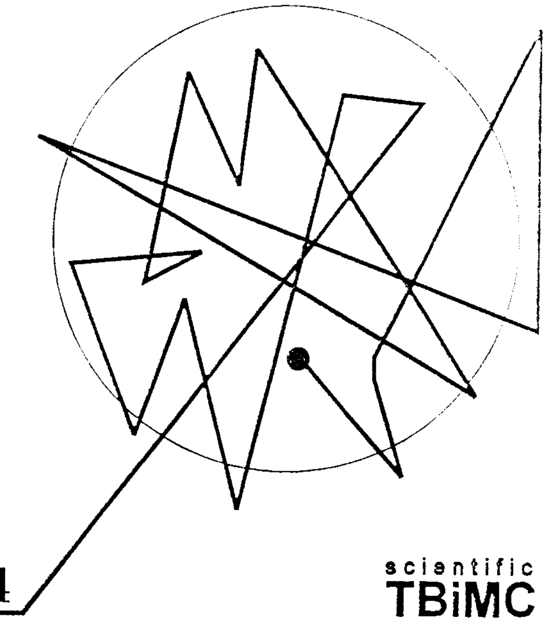
ISSN 0321-3900

Theory of Stochastic Processes



5(21) • 1999 № 3-4

scientific TBiMC publishers

Proceedings

of the

Third Ukrainian-Scandinavian Conference

in Probability Theory and Mathematical Statistics

8-12 June 1999

Kyiv, Ukraine

Theory of Stochastic Processes
Vol. 5 (21), no. 3–4, 1999, pp. 91–104

UDC 519.21

OLEG KLESOV, ISTVÁN FAZEKAS, CSABA NOSZÁLY, AND TIBOR TÓMÁCS

STRONG LAWS OF LARGE NUMBERS FOR SEQUENCES AND FIELDS

A general method to obtain strong laws of large numbers is considered. The method is extended to random fields. Several applications for dependent summands are given.

CONTENTS

0. INTRODUCTION

In this paper, we show that a Hájek–Rényi type inequality is a consequence of an appropriate maximal inequality for cumulative sums and that the latter automatically implies the strong law of large numbers (SLLN). The most important is that we made no restriction on the dependence structure of random variables. Several examples of applications are given. We do not consider independent, orthogonal and stationary sequences because results in these cases are well-known, however these are possible applications of our general approach. We concentrate on superadditive moment structures, mixingales and logarithmically weighted sums. Several other examples are given in Fazekas and Klesov (1998).

The method is extended for random fields. The same type of applications are given as for sequences. We remark that in the case of sequences our aim is to give unified simple proofs of SLLN's. For fields, however, we obtain new results in the case of mixingales and logarithmically weighted sums. A detailed discussion and several other examples are given in Noszály and Tómács (1999).

1. THE GENERAL SLLN

We shall use the following notation. $X_1, X_2, \dots$ will denote a sequence of random variables defined on a fixed probability space. The partial sums of the random variables will be $S_n = \sum_{i=1}^n X_i$ for $n \geq 1$ and $S_0 = 0$. A sequence $\{b_n\}$ will called nondecreasing if $b_i \leq b_{i+1}$ for $i \geq 1$. In the following $\mathbb{N}_0$ and $\mathbb{N}$ denote the set of nonnegative and positive integers, respectively. $\mathbb{Z}$ denotes the set of all integers.

Our first theorem is a Hájek-Rényi type maximal inequality.

1991 *AMS Mathematics Subject Classification.* 60F15.

Key words and phrases. Strong law of large numbers, maximal inequality, random field.
Supported by the Hungarian Ministry of Culture and Education under Grant No. FKFP 0121/1999

Theorem 1.1. *Let $\beta_1, \dots, \beta_n$ be a nondecreasing sequence of positive numbers. Let $\alpha_1, \dots, \alpha_n$ be non-negative numbers. Let r be a fixed positive number. Assume that for each m with $1 \le m \le n$*

$$\mathsf{E}\left[\max_{1\le l\le m} |S_l|\right]^r \le \sum_{l=1}^{m} \alpha_l. \tag{1.1}$$

Then

$$\mathsf{E}\left[\max_{1\le l\le n} \left|\frac{S_l}{\beta_l}\right|\right]^r \le 4\sum_{l=1}^{n} \frac{\alpha_l}{\beta_l^r}. \tag{1.2}$$

Proof. We can suppose that $\beta_1 = 1$. Let $c > 1$. Consider the following sets

$$A_i = \{k : c^i \le \beta_k < c^{i+1}\}, \qquad i = 0, 1, 2, \dots .$$

Grouping summands in (1.2) according to A_i's, elementary considerations give the result. For details see Theorem 1.1 of Fazekas and Klesov (1998) and Theorem 3.1 of this paper. □

Our general strong law of large numbers is the following.

Theorem 1.2. *Let $b_1, b_2, \dots$ be a nondecreasing unbounded sequence of positive numbers. Let $\alpha_1, \alpha_2, \dots$ be non-negative numbers. Let r be a fixed positive number. Assume that for each $n \ge 1$*

$$\mathsf{E}\left[\max_{1\le l\le n} |S_l|\right]^r \le \sum_{l=1}^{n} \alpha_l. \tag{1.3}$$

If

$$\sum_{l=1}^{\infty} \frac{\alpha_l}{b_l^r} < \infty \tag{1.4}$$

then

$$\lim_{n\to\infty} \frac{S_n}{b_n} = 0 \quad a.s. \tag{1.5}$$

Proof. Let $\{\beta_n\}$ be a sequence satisfying properties given in Lemma 1.2 below. Then Theorem 1.1 implies that (1.2) is satisfied. Therefore

$$\mathsf{E}\left[\sup_{l\ge 1} \left|\frac{S_l}{\beta_l}\right|\right]^r \le 4\sum_{l=1}^{\infty} \frac{\alpha_l}{\beta_l^r} < \infty.$$

This implies

$$\sup_{l\ge 1} \left|\frac{S_l}{\beta_l}\right| < \infty \qquad \text{a.s.}$$

Finally,

$$0 \le \left|\frac{S_l}{b_l}\right| = \left|\frac{S_l}{\beta_l}\right| \frac{\beta_l}{b_l} \le \left\{\sup_{l\ge 1} \left|\frac{S_l}{\beta_l}\right|\right\} \frac{\beta_l}{b_l} \to 0,$$

a.s. as $l \to \infty$. □

The following remark is often useful to check assumption (1.4) in Theorem 1.2.

Remark 1.1. *Let r be a fixed positive number. Let $b_n^r = n^\delta$, $n = 1, 2, \dots$, where $\delta > 0$. Let $\alpha_1, \alpha_2, \dots$ be non-negative numbers, $\Lambda_k = \alpha_1 + \dots + \alpha_k$ for $k \ge 1$. If*

$$\sum_{l=1}^{\infty} \Lambda_l \left(\frac{1}{b_l^r} - \frac{1}{b_{l+1}^r}\right) < \infty, \tag{1.6}$$

then (1.4) is fulfilled. □

Lemma 1.1. *Let $\{\lambda_k\}$ be a sequence of nonnegative numbers, with $\sum_{k=1}^{\infty} \frac{\lambda_k}{2^k} < \infty$. Then there exists a sequence $\{\gamma_k\}$ such that*

(i) $\gamma_k \leq \gamma_{k+1}$, $k = 1, 2, \ldots$,
(ii) $\lim_{k\to\infty} \gamma_k = \infty$,
(iii) $\sum_{k=1}^{\infty} \frac{\lambda_k}{\gamma_k} < \infty$,
(iv) $\lim_{k\to\infty} \frac{\gamma_k}{2^k} = 0$.

Proof. If finitely many λ_k's are positive then the statement is obvious. Suppose that there are infinitely many positive λ_k. Let $z = \sum_{k=1}^{\infty} \frac{\lambda_k}{2^k}$ and for $i = 0, 1, \ldots$, let n_i be the smallest integer such that $\sum_{k=n_i}^{\infty} \frac{\lambda_k}{2^k} \leq \frac{z}{2^i}$. Set $A_i = \{k : n_i \leq k < n_{i+1}\}$ then delete the emty sets A_i and reenumerate the sequence $\{A_i\}$ so that the indices in the sequence of the non-empty sets $\{A_i\}$ be $0, 1, 2, \ldots$. Let $\gamma_k = 2^{k-i/2}$ for $k \in A_i$. Then property (iv) is obvious. Property (iii) follows from

$$\sum_{k=1}^{\infty} \frac{\lambda_k}{\gamma_k} = \sum_{i=1}^{\infty} \sum_{k\in A_i} \frac{\lambda_k}{\gamma_k} \leq \sum_{i=1}^{\infty} 2^{i/2} \sum_{k\geq n_i} \frac{\lambda_k}{2^k} \leq z \sum_{i=1}^{\infty} 2^{-i/2} < \infty.$$

Property (i) has to be verified only for $k = n_{i+1} - 1, i = 1, 2, \ldots$. In this case $\gamma_{k+1}/\gamma_k = \sqrt{2}$ so (i) follows. This equality and the definition of γ_k imply (ii) as well. □

Lemma 1.2. *Let $\{b_k\}$ be a nondecreasing unbounded sequence of positive numbers. Let $\{\alpha_k\}$ be a sequence of nonnegative numbers, with $\sum_{k=1}^{\infty} \frac{\alpha_k}{b_k^r} < \infty$, where $r > 0$. Then there exists a sequence $\{\beta_k\}$ of positive numbers such that*

(a) $\beta_k \leq \beta_{k+1}$, $k = 1, 2, \ldots$,
(b) $\lim_{k\to\infty} \beta_k = \infty$,
(c) $\sum_{k=1}^{\infty} \frac{\alpha_k}{\beta_k^r} < \infty$,
(d) $\lim_{k\to\infty} \frac{\beta_k}{b_k} = 0$.

Proof. It is enough to prove for $r = 1$. Define a sequence $\{m_i\}$ of positive integers as follows

$$m_i = \max\{m : b_m \leq 2^i\}, \qquad i = 1, 2, \ldots, \qquad m_0 = 0,$$

and set $B_i = \{m_{i-1} + 1, \ldots, m_i\}$, for $i = 1, 2, \ldots$. Let $\lambda_i = \sum_{k\in B_i} \alpha_k$. Since

$$\sum_{k=1}^{\infty} \frac{\alpha_k}{b_k} = \sum_{i=1}^{\infty} \sum_{k\in B_i} \frac{\alpha_k}{b_k} \geq \sum_{i=1}^{\infty} \frac{\lambda_i}{2^i}$$

we get that $\sum_{i=1}^{\infty} \frac{\lambda_i}{2^i} < \infty$. According to Lemma 1.1, there exists a sequence $\{\gamma_k\}$ satisfying conditions (i)–(iv) of that lemma. Now we put $\beta_k = \gamma_i$, for $k \in B_i$. Properties (a), (b) and (d) are obvious. Since

$$\infty > \sum_{i=1}^{\infty} \frac{\lambda_i}{\gamma_i} = \sum_{i=1}^{\infty} \sum_{k\in B_i} \frac{\alpha_k}{\beta_k},$$

we get property (c). □

2. Applications

a. Sequences with superadditive moment function

A sequence of random variables $\{X_n, n \geq 1\}$ is said to have the r-th ($r > 0$) moment function of superadditive structure if there exists a nonnegative function $g(i, j)$ of two arguments such that for all $b \geq 0$ and $1 \leq k < k + l$

$$g(b, k) + g(b + k, l) \leq g(b, k + l)$$

and for some $\alpha > 1$

$$\mathsf{E}\,|S_{b,n}|^r \le g^\alpha(b,n), \tag{2.1}$$

where $S_{b,n} = \sum_{\nu=b+1}^{b+n} X_\nu$. One of possible choices for $g(i,j)$ is $g(i,j) = \sum_{\nu=i+1}^{i+j} a_\nu^2$.

Under the superadditivity property Móricz (1976) proved that there exists a constant $A_{r,\alpha}$ depending only on r and α such that

$$\mathsf{E}\left[\max_{k\le n}|S_k|\right]^r \le A_{r,\alpha} g_n^\alpha,$$

where $g_n = g(1,n)$. As g_n increases so we may apply Theorem 1.2. For instance, put $a_1 = g_1^\alpha$ and $a_k = g_k^\alpha - g_{k-1}^\alpha$, $k > 1$. Theorem 1.2 implies for any nondecreasing and unbounded sequence $\{b_n, n \ge 1\}$ that

$$\lim_{n\to\infty} \frac{S_n}{b_n} = 0 \qquad \text{a.s.}$$

provided

$$\sum_{n=1}^{\infty} \frac{g_n^\alpha - g_{n-1}^\alpha}{b_n^r} < \infty. \tag{2.2}$$

We also may give Marcinkiewicz–Zygmund SLLN for sequences with moment function of superadditive structure.

Theorem 2.1. *Assume that a sequence of random variables $\{X_n, n \ge 1\}$ has r-th moment function of superadditive structure with $r > 0$, $\alpha > 1$. Let $q > 0$. If*

$$\sum_{n=1}^{\infty} \frac{g_n^\alpha}{n^{1+\frac{r}{q}}} < \infty \tag{2.3}$$

then

$$\lim_{n\to\infty} \frac{S_n}{n^{1/q}} = 0 \qquad a.s.$$

Proof. By Remark 1.1, condition (2.3) is sufficient for (2.2) when $b_n = n^{1/q}$. □

b. Mixingales

Introduced by McLeish (1975) mixingales have been investigated by several authors. For example, Andrews (1988) and Hansen (1991) studied L^r ($r \ge 1$) mixingales defined as follows. Let $\{\mathfrak{F}_n, n \in \mathbb{Z}\}$ be a nondecreasing sequence of sub σ-fields and $\{X_n, n \ge 1\}$ be a sequence of random variables. Put $\mathsf{E}_m X_i = \mathsf{E}\left(X_i \middle| \mathfrak{F}_m\right)$. A sequence $(X_n, \mathfrak{F}_n)$ is an L^r mixingale if there exist nonnegative constants $\{c_i, i \ge 0\}$ and $\{\psi_m, m \ge 0\}$ such that $\psi_m \downarrow 0$ and for all $i \ge 0$ and $m \ge 0$ we have

(i) $\|\mathsf{E}_{i-m} X_i\|_r \le c_i \psi_m$,
(ii) $\|X_i - \mathsf{E}_{i+m} X_i\|_r \le c_i \psi_{m+1}$,

where $\|\xi\|_r = (\mathsf{E}\,|\xi|^r)^{1/r}$ for any random variable ξ. If $(X_n, \mathfrak{F}_n)$ is an L^r mixingale then it is an L^s mixingale for $1 \le s \le r$. Hansen (1991) got under condition $\sum_{m=1}^\infty \psi_m < \infty$ that for $r \ge 2$

$$\mathsf{E}\left[\max_{k\le n}|S_k|\right]^r \le O(1)\left(\sum_{i=1}^{n} c_i^2\right)^{r/2}. \tag{2.4}$$

Using this inequality, Hansen (1991) obtains the SLLN:

$$\lim_{n\to\infty} \frac{S_n}{n} = 0 \qquad \text{a.s.}$$

provided $\sum c_k^2/k^2 < \infty$ and $r \ge 2$. It is clear that the above SLLN is an obvious corollary of inequality (2.4) with $r = 2$ and Theorem 1.2. On the other hand, Theorem 1.2 allows one to obtain the following result.

Theorem 2.2. *Let $\{X_n, n \geq 1\}$ be an L^r mixingale, $r > 2$, and let $0 < q < 2$. If $\sum_{m=1}^{\infty} \psi_m < \infty$ and*

$$\sum_{k=1}^{\infty} \frac{1}{k^{1+\frac{r}{q}}} \left(\sum_{i=1}^{k} c_i^2 \right)^{r/2} < \infty, \tag{2.5}$$

then

$$\lim_{n\to\infty} \frac{S_n}{n^{1/q}} = 0 \qquad a.s.$$

Proof. By Theorem 1.2 and Remark 1.1, (2.4) and (2.5) imply the result. □

c. Logarithmically weighted sums

The theorem below is a special case of a Móri (1993) result. This SLLN is useful for proving almost sure central limit theorems (see e.g. Chuprunov and Fazekas (1999)).

Theorem 2.3. *Let for some $\beta > 0$ and $C > 0$*

$$|\,\mathsf{E}\, X_k X_l| \leq C \left(\frac{l}{k} \right)^{\beta}, \qquad l \leq k. \tag{2.6}$$

Then

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{X_k}{k} = 0 \qquad a.s. \tag{2.7}$$

Proof. Without loss of generality we may assume that $0 < \beta < 1$. Using assumption (2.6) and Lemma 2.1 below we get

$$\begin{aligned} \mathsf{E}\left| \sum_{k=i}^{j} \frac{X_k}{k} \right|^2 &\leq 2 \sum_{k=i}^{j} \sum_{l=i}^{k} \frac{1}{kl} |\,\mathsf{E}\, X_k X_l| \\ &\leq 2C \sum_{k=i}^{j} \sum_{l=i}^{k} \frac{1}{k^{1+\beta}} \frac{1}{l^{1-\beta}} \leq 2C \frac{2}{\beta} \left(\sum_{k=i}^{j} \frac{1}{k} \right)^{\gamma}, \end{aligned}$$

where $1 < \gamma < 2$. Now Theorem 1 of Longnecker and Serfling (1977) implies that

$$\mathsf{E}\left[\max_{i \leq n} \left| \sum_{k=1}^{i} \frac{X_k}{k} \right| \right]^2 \leq A_{2,\gamma} \frac{4C}{\beta} \left(\sum_{k=1}^{n} \frac{1}{k} \right)^{\gamma},$$

where $A_{2,\gamma}$ is a constant defined in Longnecker and Serfling (1977). Now we use $\sum_{k=1}^{n} \frac{1}{k} = \mathrm{O}(1) \log n$, as $n \to \infty$, to obtain

$$\mathsf{E}\left[\max_{i \leq n} \left| \sum_{k=1}^{i} \frac{X_k}{k} \right| \right]^2 \leq \mathrm{O}(1)(\log n)^{\gamma}.$$

Set $\alpha_n = (\log(n+1))^{\gamma} - (\log n)^{\gamma}$ and note that $\lim_{n\to\infty}(\gamma n^{-1}(\log n)^{\gamma-1}/\alpha_n = 1$. Therefore

$$\sum_{n=1}^{\infty} \frac{\alpha_n}{(\log n)^2} < \infty.$$

Therefore, Theorem 1.2 implies (2.7). □

Lemma 2.1. *Set for $i \le j$*

$$g(i,j) = \sum_{k=i}^{j} \frac{1}{k}.$$

Then for any $0 < \beta < 1$ and $1 < \gamma < 2$

$$\sum_{k=i}^{j} \sum_{l=i}^{k} \frac{1}{k^{1+\beta}} \frac{1}{l^{1-\beta}} \le \frac{2}{\beta} g^{\gamma}(i,j). \qquad \square$$

3. The general SLLN for random fields

Let d be a fixed positive integer. Throughout the rest of the paper I, J, K, L, M and N denote elements of $\mathbb{N}_0^d$. If an element of $\mathbb{N}_0^d$ is denoted by a capital letter, then its coordinates are denoted by the lower case of the same letter, i.e. N always means the vector $(n_1, \dots, n_d) \in \mathbb{N}_0^d$. We also use $\mathbf{1} = (1, \dots, 1) \in \mathbb{N}^d$ and $\mathbf{0} = (0, \dots, 0) \in \mathbb{N}_0^d$. In $\mathbb{N}_0^d$ we consider the coordinate-wise partial ordering: $M \le N$ means $m_i \le n_i$, $i = 1, \dots, d$ ($M < N$ means $M \le N$ and $N \ne M$). $N \to \infty$ is interpreted as $n_i \to \infty$, $i = 1, \dots, d$, $\lim_N a_N$ is meant in this sense. In $\mathbb{N}_0^d$ the maximum is defined coordinate-wise (actually we shall use it only for rectangles). If $N = (n_1, \dots, n_d) \in \mathbb{N}_0^d$ then $\langle N \rangle = \prod_{i=1}^d n_i$.

A numerical sequence a_N, $N \in \mathbb{N}_0^d$, is called d-sequence. If a_N is a d-sequence then its difference sequence, i.e. the d-sequence b_N for which $\sum_{M \le N} b_M = a_N$, $N \in \mathbb{N}_0^d$, will be denoted by Δa_N (i.e. $\Delta a_N = b_N$).

We shall say that a d-sequence a_N is of product type if $a_N = \prod_{i=1}^d a_{n_i}^{(i)}$, where $a_{n_i}^{(i)}$ ($n_i = 0, 1, 2, \dots$) is a (single) sequence for each $i = 1, \dots, d$. Our consideration will be confined to normalizing constants of product type: b_N will always denote $b_N = \prod_{i=1}^d b_{n_i}^{(i)}$, where $b_{n_i}^{(i)}$, $n_i = 0, 1, 2, \dots$, is a nondecreasing sequence of positive numbers for each $i = 1, \dots, d$. In this case we shall say that b_N is a positive nondecreasing d-sequence of product type. Moreover, if for each $i = 1, \dots, d$ the sequence $b_{n_i}^{(i)}$ is unbounded, then b_N is called positive, nondecreasing, unbounded d-sequence of product type.

The random field will be denoted by X_N, $N \in \mathbb{N}_0^d$. S_N is the partial sum: $S_N = \sum_{M \le N} X_M$ for $N \in \mathbb{N}_0^d$. As X_N is a field with lattice indices we shall say that X_N, $N \in \mathbb{N}_0^d$, is a d-sequence of random variables (r.v.'s). Remark that a sum or a maximum over the empty set will be interpreted as zero (i.e. $\sum_{N \in \mathcal{H}} X_N = \max_{N \in \mathcal{H}} X_N = 0$ if $\mathcal{H} = \emptyset$). As usual, $\log^+(x) = \max\{1, \log(x)\}$, $x > 0$.

The theorem below is a straightforward generalization of Theorem 1.1. Note that there are several other ways to obtain maximal inequalities of this type: see for example Klesov (1980).

Theorem 3.1. *(Hájek-Rényi type maximal inequality.) Let $N \in \mathbb{N}^d$ be fixed. Let r be a positive real number, a_M be a nonnegative d-sequence. Suppose that b_M is a positive, nondecreasing d-sequence of product type. Then*

$$\mathbb{E}\Big\{\max_{L \le M} |S_L|^r\Big\} \le \sum_{L \le M} a_L \qquad \forall M \le N$$

implies

$$\mathbb{E}\left\{\max_{M \le N} \left|\frac{S_M}{b_M}\right|^r\right\} \le 4^d \sum_{M \le N} \frac{a_M}{b_M^r}.$$

Proof. Without loss of generality we can assume that $b_{\mathbf{1}} = 1$. Fix an $N \in \mathbb{N}^d$ and for a moment a real number $c > 1$. For $I = (i_1, \dots, i_d) \in \mathbb{N}_0^d$ let us define the set

$$\mathcal{A}_I = \{\ J \in \mathbb{N}^d\ :\ J \le N \text{ and } c^{i_k} \le b_{j_k}^{(k)} < c^{i_k+1},\ k = 1, \dots, d\ \}.$$

Now we can form

$$D_I = \sum_{J \in \mathcal{A}_I} a_J \quad \text{and} \quad K = \max\{ I : \mathcal{A}_I \neq \emptyset \}.$$

If $\mathcal{A}_I \neq \emptyset$ let

$$M_I = \max\{ J : J \in \mathcal{A}_I \}$$

otherwise set $M_I = 0$. Since $\bigcup_{I \leq K} \mathcal{A}_I$ covers the rectangle $\{M \in \mathbb{N}^d : M \leq N\}$ so

$$\mathbb{E}\left\{ \max_{M \leq N} \left| \frac{S_M}{b_M} \right|^r \right\} \leq \sum_{J \leq K} \mathbb{E}\left\{ \max_{I \in \mathcal{A}_J} \left| \frac{S_I}{b_I} \right|^r \right\}.$$

By the definition of $\mathcal{A}_I$, M_I and D_I we get

$$\sum_{J \leq K} \mathbb{E}\left\{ \max_{I \in \mathcal{A}_J} \left| \frac{S_I}{b_I} \right|^r \right\} \leq \sum_{J \leq K} \left\{ \prod_{m=1}^{d} c^{-rj_m} \right\} \mathbb{E}\left\{ \max_{I \in \mathcal{A}_J} |S_I|^r \right\} \leq$$

$$\sum_{J \leq K} \left\{ \prod_{m=1}^{d} c^{-rj_m} \right\} \mathbb{E}\left\{ \max_{I \leq M_J} |S_I|^r \right\} \leq \sum_{J \leq K} \left\{ \prod_{m=1}^{d} c^{-rj_m} \right\} \sum_{I \leq M_J} a_I \leq$$

$$\sum_{J \leq K} \left\{ \prod_{m=1}^{d} c^{-rj_m} \right\} \sum_{I \leq J} D_I \leq \sum_{I \leq K} D_I \sum_{I \leq J \leq K} \left\{ \prod_{m=1}^{d} c^{-rj_m} \right\} \leq$$

$$\sum_{I \leq K} D_I \prod_{m=1}^{d} \left\{ \sum_{j=i_m}^{k_m} c^{-rj} \right\} \leq \sum_{I \leq K} D_I \prod_{m=1}^{d} \frac{c^{-ri_m}}{1 - c^{-r}} \leq$$

$$\left\{ \frac{c^r}{1 - c^{-r}} \right\}^d \sum_{I \leq K} D_I \prod_{m=1}^{d} c^{-r(i_m+1)} \leq$$

$$\left\{ \frac{c^r}{1 - c^{-r}} \right\}^d \sum_{I \leq K} \left\{ \sum_{J \in \mathcal{A}_I} a_J \right\} \prod_{m=1}^{d} c^{-r(i_m+1)} \leq$$

$$\left\{ \frac{c^r}{1 - c^{-r}} \right\}^d \sum_{I \leq K} \sum_{J \in \mathcal{A}_I} \frac{a_J}{b_J^r} \leq \left\{ \frac{c^r}{1 - c^{-r}} \right\}^d \sum_{J \leq N} \frac{a_J}{b_J^r}.$$

This proves the proposition because $\inf_{c>1} \frac{c^r}{1-c^{-r}} = 4$. □

To prove our SLLN (i.e. Theorem 3.2) we shall use Theorem 3.1 and the following lemma.

Lemma 3.1. *Let a_N be a nonnegative d-sequence and let b_N be a positive, nondecreasing, unbounded d-sequence of product type. Suppose that $\sum_N \frac{a_N}{b_N^r} < +\infty$ with a fixed real $r > 0$. Then there exists a positive, nondecreasing, unbounded d-sequence β_N of product type for which*

$$\lim_N \frac{\beta_N}{b_N} = 0 \quad \text{and} \quad \sum_N \frac{a_N}{\beta_N^r} < +\infty.$$

Proof. It is enough to prove for $r = 1$. If $d = 1$ then our proposition is Lemma 1.2. Let $d \geq 2$. Then

$$+\infty > \sum_N \frac{a_N}{b_N} = \sum_{n_1} \frac{1}{b_{n_1}^{(1)}} \sum_{n_2,\ldots,n_d} \frac{a_N}{\prod_{m=2}^{d} b_{n_m}^{(m)}} = \sum_{n_1} \frac{1}{b_{n_1}^{(1)}} T_{n_1}$$

with $T_{n_1} = \sum_{n_2,\dots,n_d} \frac{a_N}{\prod_{m=2}^d b_{n_m}^{(m)}}$. Applying Lemma 1.2, we get that there exists an unbounded, positive, nondecreasing sequence $\beta_n^{(1)}$ so that

$$\lim_n \frac{\beta_n^{(1)}}{b_n^{(1)}} = 0 \quad \text{and} \quad \sum_{n_1} \frac{1}{\beta_{n_1}^{(1)}} T_{n_1} < +\infty.$$

If we have already obtained $\beta_n^{(m)}$ for $m = 1, \dots, k$, $k < d$, then replacing in the above procedure b_N by $\prod_{m=1}^k \beta_{n_m}^{(m)} \prod_{l=k+1}^d b_{n_l}^{(l)}$ and coordinate 1 by coordinate $k+1$, we get an appropriate $\beta_n^{(k+1)}$. Finally, by setting $\beta_N = \prod_{m=1}^d \beta_{n_m}^{(m)}$, it obviously satisfies the requirements. □

The following theorem is an extension of Theorem 1.2.

Theorem 3.2. *Let a_N, b_N be nonnegative d-sequences and let $r > 0$. Suppose that b_N is a positive, nondecreasing, unbounded d-sequence of product type. Then*

$$\sum_N \frac{a_N}{b_N^r} < +\infty$$

and

$$\mathbb{E}\Big\{\max_{M \le N} |S_M|^r\Big\} \le \sum_{M \le N} a_M \qquad \forall\, N \in \mathbb{N}^d$$

imply

$$\lim_N \frac{S_N}{b_N} = 0 \quad a.s.$$

Proof. Let β_N be the sequence obtained in the previous lemma. According to Theorem 3.1:

$$\mathbb{E}\Big\{\max_{M \le N} \Big|\frac{S_M}{\beta_M}\Big|^r\Big\} \le 4^d \sum_{M \le N} \frac{a_M}{\beta_M^r} \quad \forall N \in \mathbb{N}^d.$$

Hence

$$\mathbb{E}\Big\{\sup_N \Big|\frac{S_N}{\beta_N}\Big|^r\Big\} \le 4^d \sum_N \frac{a_N}{\beta_N^r}.$$

Therefore

$$\sup_N \Big|\frac{S_N}{\beta_N}\Big|^r < +\infty \quad \text{a.s.}$$

Now

$$\Big|\frac{S_N}{b_N}\Big| = \frac{\beta_N}{b_N}\Big|\frac{S_N}{\beta_N}\Big| \le \frac{\beta_N}{b_N} \sup_K \Big|\frac{S_K}{\beta_K}\Big|$$

proves the theorem because $\lim_N \frac{\beta_N}{b_N} = 0$. □

4. Applications for random fields

a. Fields with superadditive moment structures

Definition 4.1. *A d-sequence of random variables X_N is said to have r-th moment function of superadditive structure (MFSS) if*

$$\mathbb{E}\Big\{\Big|\sum_{I \le K \le J} X_K\Big|^r\Big\} \le g(I,J)^\alpha \quad \forall I, J \in \mathbb{N}^d, \qquad (MFSS)$$

where g is superadditive on $\mathbb{N}^d \times \mathbb{N}^d$, $r > 0$ and $\alpha > 1$. A function g on $\mathbb{N}^d \times \mathbb{N}^d$ is said to be superadditive if

$$g(I,(j_1,\dots,j_{m-1},k,j_{m+1},\dots,j_d)) + g(i_1,\dots,i_{m-1},k+1,i_{m+1},\dots,i_d),J)$$

can be majorized by $g(I, J)$ for any $m = 1, \ldots, d$ and for any $i_m \le k < j_m$. □

Remark that the notion of r-th MFSS was used by Móricz (1976). We shall apply the following result of Móricz (see Móricz (1983), Corollary 1 or Móricz (1977), Theorem 7).

Remark 4.1. *Let $r \ge 1, \alpha > 1$ and let g be nonnegative, superadditive function on $\mathbb{N}^d \times \mathbb{N}^d$. Let X_N be a d-sequence of random variables such that (MFSS) is satisfied. Then there is a constant $A_{r,\alpha,d}$ for which*

$$\mathbb{E}\Big\{\max_{K\le N}|S_K|^r\Big\} \le A_{r,\alpha,d}\, g(1,N)^\alpha \quad \forall N \in \mathbb{N}^d. \qquad \square$$

One can easily verify that the above proposition is true in the case of $0 < r < 1$, too.

We prove a Marcinkiewicz-Zygmund type SLLN for d-sequences with superadditive moment structure. Theorem 4.1 is a generalization of Theorem 2.1. For the sake of completness we start with a simple technical lemma on partial summation.

Lemma 4.1. *Let a_N, b_N be nonnegative d-sequences such that $b_N = \frac{1}{\langle N\rangle^\alpha}$ for some $\alpha > 0$. Then*

$$\sum_N (-1)^d \Lambda_N \Delta b_{N+1} < +\infty$$

implies

$$\sum_N a_N b_N < +\infty,$$

where $\Lambda_N = \sum_{M\le N} a_M$. □

Theorem 4.1. *Let $r > 0$, $\alpha > 1$ and suppose that X_N has r-th MFSS and $\Delta g(1, N)^\alpha$ is nonnegative for any $N \in \mathbb{N}^d$. Then for arbitrary $q > 0$*

$$\sum_N \frac{g(1,N)^\alpha}{\langle N\rangle^{1+\frac{r}{q}}} < +\infty \tag{4.1}$$

implies

$$\lim \frac{S_N}{\langle N\rangle^{\frac{1}{q}}} = 0 \quad a.s.$$

Proof. Using Remark 4.1 we get for all $N \in \mathbb{N}^d$ that:

$$\mathbb{E}\Big\{\max_{M\le N}|S_M|^r\Big\} \le A_{r,\alpha,d}\, g(1,N)^\alpha.$$

Let $b_N = \frac{1}{\langle N\rangle^{\frac{r}{q}}}$. Since $\prod_{m=1}^d \Big\{\frac{1}{n_m^{\frac{r}{q}}} - \frac{1}{(n_m+1)^{\frac{r}{q}}}\Big\} \le C\frac{1}{\langle N\rangle^{1+\frac{r}{q}}}$ for some $C > 0$, so (4.1) implies

$$\sum_N (-1)^d g(1,N)^\alpha \Delta b_{N+1} < +\infty.$$

Finally, we apply Lemma 4.1 and Theorem 3.2 to obtain the result. □

b. Logarithmically weighted sums

In the lemma below $[x]$, $x \ge 0$ denotes the integer part of x, i.e. [x] is the largest integer for which $[x] \le x$.

Lemma 4.2. *(a) Let $n \in \mathbb{N}$ and $0 < \beta < 1$. Then there is a constant $C_{d,\beta}$ depending only on d and β such that:*

$$\sum_{m_1=1}^{n}\sum_{m_2=1}^{\left[\frac{n}{m_1}\right]}\cdots\sum_{m_d=1}^{\left[\frac{n}{m_1 m_2\cdots m_{d-1}}\right]} \frac{1}{\langle M\rangle^{1-\beta}} \le C_{d,\beta}\, n^\beta (\log^+ n)^{d-1}.$$

(b) Let $0 < \beta < 1$, $1 < \gamma < 2$ and $I, M, J \in \mathbb{N}^d$, $I \le M \le J$. Then there is a constant $C_{d,\beta}$ depending only on d and β such that:

$$\sum_{I\le M\le J} \sum_{\substack{I\le K\le J\\ \langle K\rangle\le\langle M\rangle}} \frac{1}{\langle M\rangle^{1+\beta}} \frac{1}{(\log^+\langle M\rangle)^{d-1}} \frac{1}{\langle K\rangle^{1-\beta}} \le C_{d,\beta}\Big\{\sum_{I\le M\le J}\frac{1}{\langle M\rangle}\Big\}^{\gamma}.$$

Proof. (a) The case $d = 1$ is well known from elementary analysis. We prove by induction on d. Suppose that the statement is true for $d = f$. Let $n \in \mathbb{N}$ and $0 < \beta < 1$. Then

$$\sum_{m_1=1}^{n} \sum_{m_2=1}^{\left[\frac{n}{m_1}\right]} \cdots \sum_{m_{f+1}=1}^{\left[\frac{n}{m_1 m_2\cdots m_f}\right]} \frac{1}{\langle M\rangle^{1-\beta}} =$$

$$\sum_{m_1=1}^{n} \frac{1}{m_1^{1-\beta}} \sum_{m_2=1}^{\left[\frac{n}{m_1}\right]} \cdots \sum_{m_{f+1}=1}^{\left[\frac{n}{m_1 m_2\cdots m_f}\right]} \frac{1}{(m_2\cdots m_f)^{1-\beta}}.$$

Now applying the hypothesis for $\left[\frac{n}{m_1}\right]$ we get that the above expression is majorized by:

$$C_{f,\beta} \sum_{m_1=1}^{n} \frac{1}{m_1^{1-\beta}} \left[\frac{n}{m_1}\right]^{\beta} \left\{\log^+\left[\frac{n}{m_1}\right]\right\}^{f-1} \le C_{f,\beta}\, n^{\beta} (\log^+ n)^{f-1} \sum_{m_1=1}^{n} \frac{1}{m_1} \le$$

$$C_{f,\beta}\, n^{\beta}(\log^+ n)^{f-1} C \log^+ n$$

with certain $C > 0$ (here we used the fact $\left[\frac{1}{c}\left[\frac{a}{b}\right]\right] = \left[\frac{a}{bc}\right]$ for $a, b, c \in \mathbb{N}$).

(b) In the case $\sum_{I\le M\le J} \frac{1}{\langle M\rangle} \le 1$ we get that

$$\sum_{I\le M\le J} \sum_{\substack{I\le K\le J\\ \langle K\rangle\le\langle M\rangle}} \frac{1}{\langle M\rangle^{1+\beta}(\log^+\langle M\rangle)^{d-1}\langle K\rangle^{1-\beta}} \le$$

$$\sum_{I\le M\le J} \sum_{\substack{I\le K\le J\\ \langle K\rangle\le\langle M\rangle}} \frac{1}{\langle M\rangle^{1+\beta}\langle K\rangle^{1-\beta}} \le$$

$$\sum_{I\le M\le J} \sum_{\substack{I\le K\le J\\ \langle K\rangle\le\langle M\rangle}} \frac{1}{\langle M\rangle\langle K\rangle} \le \Big\{\sum_{I\le M\le J}\frac{1}{\langle M\rangle}\Big\}^{2} \le \Big\{\sum_{I\le M\le J}\frac{1}{\langle M\rangle}\Big\}^{\gamma}.$$

In the case $\sum_{I\le M\le J} \frac{1}{\langle M\rangle} > 1$ using part (a) and the simple fact

$$\sum_{m_1=1}^{n} \sum_{m_2=1}^{\left[\frac{n}{m_1}\right]} \cdots \sum_{m_d=1}^{\left[\frac{n}{m_1 m_2\cdots m_{d-1}}\right]} \frac{1}{\langle M\rangle^{1-\beta}} = \sum_{\substack{M\in\mathbb{N}^d\\ \langle M\rangle\le n}} \frac{1}{\langle M\rangle^{1-\beta}}, \qquad n \in \mathbb{N},$$

we get that

$$\sum_{I\le M\le J} \frac{1}{\langle M\rangle^{1+\beta}(\log^+\langle M\rangle)^{d-1}} \sum_{\substack{I\le K\le J\\ \langle K\rangle\le\langle M\rangle}} \frac{1}{\langle K\rangle^{1-\beta}} \le$$

$$C_{d,\beta} \sum_{I\le M\le J} \frac{1}{\langle M\rangle^{1+\beta}(\log^+\langle M\rangle)^{d-1}} \langle M\rangle^{\beta}(\log^+\langle M\rangle)^{d-1} =$$

$$C_{d,\beta} \sum_{I\le M\le J} \frac{1}{\langle M\rangle} \le C_{d,\beta}\Big\{\sum_{I\le M\le J}\frac{1}{\langle M\rangle}\Big\}^{\gamma}. \quad \square$$

Theorem 4.2. *Let $X_N, N \in \mathbb{N}^d$, be a d-sequence of random variables and suppose that for some $C > 0,\ \beta > 0$*

$$|\mathbb{E}(X_K X_L)| \le C\left\{\frac{\langle K\rangle}{\langle L\rangle}\right\}^\beta \frac{1}{(\log^+\langle L\rangle)^{d-1}} \quad \textit{if } \langle K\rangle \le \langle L\rangle.$$

Then

$$\lim_N \frac{1}{\prod_{i=1}^d \log^+ n_i} \sum_{K\le N} \frac{X_K}{\langle K\rangle} = 0 \quad a.s.$$

Proof. It is enough to prove for $0 < \beta < 1$. Let $I \le J$. Then

$$\mathbb{E}\left\{\left|\sum_{I\le K\le J} \frac{X_K}{\langle K\rangle}\right|^2\right\} \le 2 \sum_{I\le L\le J} \sum_{\substack{I\le K\le J \\ \langle K\rangle\le\langle L\rangle}} \frac{1}{\langle K\rangle\langle L\rangle}|\mathbb{E}(X_K X_L)| \le$$

$$2C \sum_{I\le L\le J} \sum_{\substack{I\le K\le J \\ \langle K\rangle\le\langle L\rangle}} \frac{1}{\langle K\rangle^{1-\beta}\langle L\rangle^{1+\beta}(\log^+\langle L\rangle)^{d-1}}.$$

Let $1 < \gamma < 2$. It follows from Lemma 4.2 (b) that

$$\mathbb{E}\left\{\left|\sum_{I\le K\le J} \frac{X_K}{\langle K\rangle}\right|^2\right\} \le D_{d,\beta}\left\{\sum_{I\le L\le J} \frac{1}{\langle L\rangle}\right\}^\gamma,$$

where $D_{d,\beta} > 0$ depends only on d and β. Now, from Remark 4.1 we get that

$$\mathbb{E}\left\{\max_{I\le J}\left|\sum_{K\le I} \frac{X_K}{\langle K\rangle}\right|^2\right\} \le C_{d,\beta,\gamma}\left\{\sum_{K\le J} \frac{1}{\langle K\rangle}\right\}^\gamma \quad \forall J,$$

where $C_{d,\beta,\gamma} > 0$ depends only on d, β and γ. ¿From Jensen's inequality we have:

$$\mathbb{E}\left\{\max_{I\le J}\left|\sum_{K\le I} \frac{X_K}{\langle K\rangle}\right|^{\frac{2}{\gamma}}\right\} \le (C_{d,\beta,\gamma})^{\frac{1}{\gamma}} \sum_{K\le J} \frac{1}{\langle K\rangle} \quad \forall J.$$

Now we can apply Theorem 3.2 because

$$\sum_N \frac{1}{(\prod_{m=1}^d \log n_m)^{\frac{2}{\gamma}}} \frac{1}{\langle N\rangle} < +\infty. \qquad \square$$

c. Mixingales

Now we define multiindex L^r mixingales and prove an SLLN for a special class of such mixingales.

Let X_N and $\mathcal{A}_N$ be a d-sequence of random variables and be a d-sequence of σ-subalgebras, respectively. We shall say that the pair $(X_N, \mathcal{A}_N)$ has property (ex) if

$$\mathbb{E}\Big(\mathbb{E}(X_L|\mathcal{A}_M)|\mathcal{A}_N\Big) = \mathbb{E}\Big(X_L|\mathcal{A}_{\min(M,N)}\Big) \qquad L, M, N \in \mathbb{Z}^d. \qquad (ex)$$

This property is widely used in the theory of multiindex martingales (see e.g. Fazekas (1983)).

Definition 4.2. *Let $r \geq 1$, $(\Omega, \mathcal{A}, P)$ be a probability space, X_N be a d-sequence of random variables with finite r-th moment, $\mathcal{A}_N$ $(N \in \mathbb{Z}^d)$ be a nondecreasing d-sequence of σ-subalgebras of $\mathcal{A}$. The pair $(X_N, \mathcal{A}_M)$ is called L^r-mixingale if*

$$\text{(a)} \qquad \left\|\mathbb{E}(X_N|\mathcal{A}_{N-M})\right\|_r \leq c_N \Psi_{-M} \quad \textit{if} \ m_i \geq 0 \ \textit{for some} \ i = 1, \ldots, d,$$

$$\text{(b)} \qquad \left\|X_N - \mathbb{E}(X_N|\mathcal{A}_{N+M})\right\|_r \leq c_N \Psi_M \quad \textit{if} \ M \geq 0,$$

where c_N $(N \in \mathbb{N}^d)$, Ψ_N $(N \in \mathbb{Z}^d)$ are d-sequences with $\Psi_N \to 0$ as $n_i \to -\infty$ for some $i = 1, \ldots, d$, $\Psi_N \to 0$ as $n_i \to \infty$ for each $i = 1, \ldots, d$, and there is a constant $C > 0$ for which

$$\Psi_M \leq C \Psi_N$$

for any $M, N \in \mathbb{Z}^d$ with $N - 1 \leq M \leq N$.

The following lemma is a straightforward generalization of Lemma 1 and Lemma 2 of Hansen (1991).

Lemma 4.3. *(a) Let $r \geq 2$ and $(X_N, \mathcal{A}_M)$ be an L^r mixingale having property (ex). Then there exists a constant $F_{r,d} > 0$ such that*

$$\left\| \max_{M \leq N} |S_M| \right\|_r \leq F_{r,d} \sum_{K \in \mathbb{Z}^d} \left\{ \sum_{M \leq N} \left\| X_M^{(K)} \right\|_r^2 \right\}^{\frac{1}{2}},$$

where $X_M^{(K)} = \Delta \mathbb{E}(X_M | \mathcal{A}_{M-K})$ and here the difference is taken according to the subscript of $\mathcal{A}$ while the subscript of X remains fixed.

(b) Let $r \geq 2$ and $(X_N, \mathcal{A}_M)$ be an L^r mixingale having property (ex) such that $\sum_{K \in \mathbb{Z}^d} \Psi_K < +\infty$. Then

$$\left\| \max_{M \leq N} |S_M| \right\|_r \leq C_{r,d} \left\{ \sum_{M \leq N} c_M^2 \right\}^{\frac{1}{2}}$$

for some $C_{r,d}$.

Proof. (a) Let $N, K \in \mathbb{N}^d$. Then

$$\sum_{-K \leq M \leq K} X_N^{(M)} = \sum_{-K \leq M \leq K} \Delta \mathbb{E}(X_N | \mathcal{A}_{N-M}),$$

which is equal to the difference of $\mathbb{E}(X_N|\mathcal{A}_L)$ (according the subscript L) on the rectangle $[N - K - 1, N + K]$. By the definition of the L^r-mixingale, one can see that

$$\lim_K \left\{ \sum_{-K \leq M \leq K} X_N^{(M)} - X_N \right\} = 0 \quad \text{in } L^r.$$

Hence, using the triangle inequality in L^r, we get

$$\left\| \max_{M \leq N} |S_M| \right\|_r = \left\| \max_{M \leq N} \left| \sum_{L \leq M} \sum_{K \in Z^d} X_L^{(K)} \right| \right\|_r \leq$$

$$\left\| \max_{M \leq N} \sum_{K \in Z^d} \left| \sum_{L \leq M} X_L^{(K)} \right| \right\|_r \leq \sum_{K \in Z^d} \left\| \max_{M \leq N} \left| \sum_{L \leq M} X_L^{(K)} \right| \right\|_r = (I).$$

Let $K \in \mathbb{N}^d$ be fixed. With the help of property (ex) it is easy to check that the pair $(Z_M, \mathcal{F}_M)$ is martingale difference, where

$$Z_M = X_M^{(K)} \quad \text{and} \quad \mathcal{F}_M = \mathcal{A}_{M-K-1}.$$

Hence by the Doob and the Burkholder inequalities (see e.g. Burkholder (1966), Mètraux (1978), Noszály and Tómács (1999)) and by the triangle inequality in the space $L^{\frac{r}{2}}$, we have

$$(I) \le D_{r,d} \sum_{K\in Z^d} \Big\| \sum_{L\le N} X_L^{(K)} \Big\|_r \le F_{r,d} \sum_{K\in Z^d} \Big\| \Big\{ \sum_{L\le N} \big|X_L^{(K)}\big|^2 \Big\}^{\frac{1}{2}} \Big\|_r =$$

$$F_{r,d} \sum_{K\in Z^d} \Big\| \Big\{ \sum_{L\le N} \big|X_L^{(K)}\big|^2 \Big\} \Big\|_{\frac{r}{2}}^{\frac{1}{2}} \le F_{r,d} \sum_{K\in Z^d} \Big\{ \sum_{L\le N} \Big\| \big|X_L^{(K)}\big|^2 \Big\|_{\frac{r}{2}} \Big\}^{\frac{1}{2}}.$$

(b) Let us consider $X_L^{(K)}$. If $k_m \ge 0$ for some $m = 1,\dots,d$, then

$$\|X_L^{(K)}\|_r = \Big\| \Delta \mathbb{E}(X_L | \mathcal{A}_{L-K}) \Big\|_r \le c_L 2^d C \Psi_{-K}.$$

Otherwise, if $k_m \le -1$ for each $m = 1,\dots,d$, then by Definition 4.2,

$$\Big\| \Delta \mathbb{E}(X_L|\mathcal{A}_{L-K}) \Big\|_r \le \sum_{L-K-1\le M\le L-K} \Big\| X_L - \mathbb{E}(X_L|\mathcal{A}_M) \Big\|_r \le c_L 2^d C \Psi_{-K}.$$

Hence, by part (a),

$$\Big\| \max_{M\le N} |S_M| \Big\|_r \le F_{r,d} \sum_{K\in Z^d} \Big\{ \sum_{L\le N} c_L^2 2^{2d} C^2 \Psi_{-K}^2 \Big\}^{\frac{1}{2}} =$$

$$F_{r,d} 2^d C \Big\{ \sum_{K\in Z^d} \Psi_K \Big\} \Big\{ \sum_{L\le N} c_L^2 \Big\}^{\frac{1}{2}}. \quad \square$$

Theorem 4.3. *Let $r \ge 2$ and $(X_N, \mathcal{A}_M)$ be an L^r mixingale with property (ex). Then*

$$\sum_{N\in\mathbb{Z}^d} \Psi_N < \infty \quad \text{and} \quad \sum_{N\in\mathbb{N}^d} \frac{1}{\langle N\rangle^{1+\frac{r}{q}}} \Big\{ \sum_{M\le N} c_M^2 \Big\}^{\frac{r}{2}} < \infty$$

imply

$$\lim_N \frac{S_N}{\langle N\rangle^{\frac{1}{q}}} = 0 \quad a.s.$$

provided that the d-sequence c_N is of product type.

Proof. Easy consequence of Theorem 3.2 and Lemma 4.3 (b). $\square$

Bibliography

1. D. W. K. Andrews, *Laws of large numbers for dependent non-identically distributed random variables*, Econometric Theory **4** (1988), 458–467.
2. D. L. Burkholder, *Martingale transforms*, Ann. Math. Stat. **37** (1966), 1994-1504.
3. A. N. Chuprunov and I. Fazekas, *Almost sure versions of some analogues of the invariance principle*, Publ. Math. Debrecen **54/3-4** (1999), 457-471.
4. I. Fazekas, *Convergence of vector valued martingles with multidimensional indices*, Publ. Math. Debrecen **30/1-2** (1983), 157-164.
5. I. Fazekas and O. Klesov, *A general approach to the strong law of large numbers*, Kossuth University, Debrecen, Technical Report **4** (1998) (to appear in Theory of Probability Appl).
6. J. Hájek and A. Rényi, *Generalization of an inequality of Kolmogorov*, Acta Math. Acad. Sci. Hungar. **6** (1955), no. 3–4, 281–283.
7. B. E. Hansen, *Strong laws for dependent heterogeneous processes*, Econometric Theory **7** (1991), 213–221; *Erratum*, Econometric Theory **8** (1992), 421–422.
8. O. Klesov, *The Hajek-Rényi inequality for random fields and strong law of large numbers*, Teor. Veroyatnost. i Mat. Statist. **22** (1980), 58-66. (Russian)
9. D. L. McLeish, *A maximal inequality and dependent strong laws*, Ann. Probab. **3** (1975), no. 5, 829–839.

10. M. Longnecker and R. J. Serfling, *General moment and probability inequalities for the maximum partial sum*, Acta Math. Sci. Hungar. **30** (1977), no. 1–2, 129–133.
11. C. Mètraux, *Quelques inégalités pour martingales à paramètre bidimensionnel*, Lecture Notes in Mathematics **649** (1978), Springer-Verlag, Berlin, 170-179.
12. T. F. Móri, *On the strong law of large numbers for logarithmically weighted sums*, Annales Univ. Sci. Budapest **36** (1993), 35–46.
13. F. Móricz, *Moment inequalities and the strong laws of large numbers*, Z. Wahrscheinlichkeitstheorie verw. Gebiete **35** (1976), 299–314.
14. F. Móricz, *Moment inequalities for the maximum of partial sums of random fields*, Acta Sci. Math. **39** (1977), 353-366.
15. F. Móricz, *A general moment inequality for the maximum of the rectangular partial sum of multiple series*, Acta Math. Hung. **41** (1983), 337-346.
16. Cs. Noszály and T. Tómács, *General approach to strong laws of large numbers for fields of random variables*, Kossuth University, Debrecen, Technical Report **10** (1999).

Department of Mathematics #1, National Technical University of Ukraine, pr. Peremogy, 37, Kiev 252056, Ukraine
E-mail: oleg@tbimc.freenet.kiev.ua

Institute of Mathematics and Informatics, Kossuth University, P.O. Box 12, H–4010 Debrecen, Hungary
E-mail: fazekasi@math.klte.hu

Institute of Mathematics and Informatics, Kossuth University, P.O. Box 12, H–4010 Debrecen, Hungary
E-mail: noszaly@math.klte.hu

Department of Mathematics, Eszterházy Károly Teacher's Training College, P.O. Box 43, H–3301 Eger, Hungary
E-mail: tomacs@gemini.ektf.hu